

# Galois theory for semiclones

MIKE BEHRISCH

**ABSTRACT.** We present a GALOIS theory connecting finitary operations with pairs of finitary relations one of which is contained in the other. The GALOIS closed sets on both sides are characterised as locally closed subuniverses of the full iterative function algebra (semiclones) and relation pair clones, respectively. Moreover, we describe the modified closure operators if only functions and relation pairs of a certain bounded arity, respectively, are considered.

## 1. INTRODUCTION

*Clones of operations*, i.e. composition closed sets of operations containing all projections (cf. [30, 34, 24, 15]), play an important role in universal algebra as they encode structural properties independently of the similarity type of the algebra. It is well-known (see [9, 14], translations available in [7, 8]) that on finite carrier sets clones are in a one-to-one correspondence with structures called *relational clones*. This is established via the GALOIS correspondence Pol-Inv, which is induced by the relation of “functions *preserving* relations”. In general, i.e. including in particular the case of infinite sets, so-called *local closure operators* come into play (see [14, 29, 28, 23, 3]), and also the notion of relational clone as known from finite domains needs to be generalised (cf. *ibid.*). In this way the GALOIS connection singles out certain *locally closed* clones from the lattice of all clones on a given set. These clones can also be seen as those which are topologically closed w.r.t. the topology that one gets by endowing each set  $A^{A^n}$ ,  $n \in \mathbb{N}$ , with the product topology arising from  $A$  initially carrying the discrete topology (see e.g. [5, 4]).

By equipping the set of all finitary functions on a fixed set  $A$  with a finite number of operations (including permutation of variables, identification of variables, introduction of fictitious variables, a certain binary composition operation and a projection as a constant; we present more details later on), one obtains the *full function algebra* of finitary functions on  $A$ . It is known (cf. e.g. [30, 24]) that the clones on  $A$  are exactly the carrier sets of subalgebras of this structure. This relationship is a special case of the one between the *full iterative function algebra*, also known as *iterative POST algebra* (introduced by Malcev in [26]), and its subuniverses (called *POST algebras* in [9]), which have often only been referred to as *closed classes* of functions in the Russian literature (e.g. [20, 21]). These are similar in spirit to clones, but they do not

---

*Date:* 19th September 2015.

*2010 Mathematics Subject Classification.* Primary: 08A40; Secondary: 08A02, 08A99.

*Key words and phrases.* iterative algebra, semiclone, relation pair clone, Galois theory.

Supported by the Austrian Science Fund (FWF) under grant I836-N23.

need to contain the projections (*selectors* in the terminology of [26]) as the iterative POST algebra omits the projection constant in its signature compared to the full function algebra.

In analogy to the Pol-Inv GALOIS connection, a GALOIS correspondence Polp-Invp has been developed in [16] (see also [17, 18, 19]) based on the notion of functions preserving pairs  $(\varrho, \varrho')$  of relations  $\varrho' \subseteq \varrho$ . For finite carrier sets the GALOIS closed sets have been characterised to be precisely the subuniverses of the full iterative POST algebra and the subuniverses of a suitably defined relation pair algebra, respectively. To the best knowledge of the author, a generalisation of this result to arbitrary base sets has not yet appeared in the literature. In particular the general (and thus infinite) case is also missing in Table 1 of [13, p. 296] summarising related GALOIS connections and characterisations of their closure operators.

In this article it is our aim to fill in this gap. We first coin the notion of a *semiclone*, which relates to transformation semigroups in the same way as clones relate to transformation monoids. It is not hard to figure out that semiclones and subuniverses of the full iterative POST algebra coincide. However, not only is the name shorter, but also do we feel that the way how a semiclone is defined is much more similar to the usual definition of a clone and easier to grasp than that of a subalgebra of the full iterative function algebra; hence the proposition of the new terminology of semiclones. Unfortunately, our semiclones are different from those appearing in [32], which are closed w.r.t. a different form of composition, do have to contain the identity operation, but are not necessarily closed under variable substitutions.

In a similar fashion as one needed to generalise the notion of relational clone to accommodate the closed sets of  $\text{Inv}_A \text{Pol}_A$  for infinite carrier sets  $A$ , it will be necessary to modify the relation pair algebra proposed by Harnau in [16]. We shall refer to the corresponding (new) subuniverses as *relation pair clones*.

Using the same local closure operator as introduced in [14, 28, 29] for sets of functions (the topological closure), and appropriately modifying the local closure on the side of relation pairs, we shall prove the following two main results: the GALOIS closed sets of operations w.r.t.  $\text{Polp}_A \text{Invp}_A$  are exactly the locally closed semiclones. Dually, the closed sets of  $\text{Invp}_A \text{Polp}_A$  are precisely the locally closed relation pair clones.

Because it fits nicely in this context, we shall more specifically study and characterise what it means that a semiclone can be described in the form  $\text{Polp}_A Q$  for some set  $Q$  of at most  $s$ -ary relation pairs, and that a relation pair clone is given by  $\text{Invp}_A F$  using a set  $F$  of at most  $s$ -ary operations. As in [29] this involves certain *s-local* closure operators, and, in general, the reader may find that quite a few results in our text are analogous to those in [29], where similar questions have been studied w.r.t. Pol-Inv.

We mention that a related, in some sense more general, GALOIS connection has been studied in [27] (finite case) and [12, 11]. There, for fixed sets  $A$  and  $B$ , functions  $f: A^n \rightarrow B$  have been related to pairs of relations  $R \subseteq A^m, S \subseteq B^m$  for some  $m \in \mathbb{N}_+$ , called *relational constraints*. In this situation the GALOIS closed sets on the functional side are also closed w.r.t. variable substitutions (as our semiclones), but already for syntactic reasons cannot be closed w.r.t. compositions. So even if one considers the special case that  $B = A$ , the results from [12] and [11] describe similar but differently closed sets of functions due

to other objects on the dual side (there is no containment condition for the relations as in our setting since for general  $A$  and  $B$  there cannot be one).

We acknowledge that, perhaps, it could be possible to derive our results by restricting the relational side of the GALOIS correspondence studied in [12, 11], but we think that the way of describing the closed objects on the dual side used there is (and has to be) more complicated (using so-called conjunctive minors), in fact, too technical for our situation. Besides, our strategy of proof exhibits more similarities with the classical arguments known from clones and relational clones. Also the local closures developed for relational constraints in [12, 11] necessarily need to be modified (see Remark 2.16) to be used with our relation pairs due to the inclusion requirement in their definition.

Still a different weakening of the notion of clone and an associated GALOIS theory for arbitrary domains has been considered in [25]: there sets of functions that contain (as clones do) all projections, are closed under substitution of one function into the first place of another one, permutation of positions and addition of fictitious variables but are not necessarily closed under variable identification (as semiclones are) have been characterised in terms of closed sets of so-called *clusters*. For the classes of functions characterised in [25] contain all projections, these results explore a separate direction and cannot be exploited either to obtain the missing general (infinite) case for semiclones.

**Acknowledgements.** The author expresses his gratitude to Erhard Aichinger for an invitation to the Institute for Algebra at Johannes Kepler University Linz, which enabled fruitful discussions on some aspects of the topic with members of the institute including Erhard Aichinger, Peter Mayr, Keith Kearnes and Ágnes Szendrei. The author wishes to thank them, too, for their valuable comments and contributions.

## 2. PRELIMINARIES

**2.1. Notation, functions and relations.** In this article the symbol  $\mathbb{N}$  will denote the set of all natural numbers (including zero), and  $\mathbb{N}_+$  will be used for  $\mathbb{N} \setminus \{0\}$ . Moreover, we shall make use of the standard set theoretic representation of natural numbers by JOHN VON NEUMANN, i.e.  $n = \{i \in \mathbb{N} \mid i < n\}$  for  $n \in \mathbb{N}$ . The *power set* of a set  $S$  will be denoted by  $\mathfrak{P}(S)$ .

When discussing semiclones, relation pair clones and their GALOIS theory we shall make no further assumptions on the carrier set, which we usually represent by  $A$ . Any finite (including 0) or infinite cardinality is allowed for  $A$ .

For sets  $A$  and  $B$  we write  $A^B$  for the set of all mappings from  $B$  to  $A$ . The order of composition employed in this article is from right to left, i.e.  $g \circ f \in C^A$  for  $f \in B^A$  and  $g \in C^B$ . That is,  $g \circ f$  maps elements  $a \in A$  to  $g(f(a))$ . For any index set  $I$ , sets  $A$  and  $(B_i)_{i \in I}$  and maps  $(f_i: A \rightarrow B_i)_{i \in I}$ , their *tupling* is the unique map  $h: A \rightarrow \prod_{i \in I} B_i$  satisfying  $\pi_i \circ h = f_i$  for each  $i \in I$ , where  $\pi_i: \prod_{j \in I} B_j \rightarrow B_i$  is the  $i$ -th projection map belonging to the CARTESIAN product  $\prod_{j \in I} B_j$ . As any ambiguity can usually be resolved from the context, we denote the tupling  $h$  by  $(f_i)_{i \in I}$ , in the same way as the tuple  $(f_i)_{i \in I}$ .

The notion of tupling is, of course, meaningful (by definition) in any category having suitable products, and hence the following simple lemma about

composition of tuplings can be proven in such a general context. We recall here just its instance for the category of sets (cf. [3, Lemma 2.5]):

**Lemma 2.1.** *Let  $I$  and  $J$  be arbitrary index sets,  $k, m, n \in \mathbb{N}$  natural numbers, and  $A, B, D, X$  and  $B_i$  ( $i \in I$ ),  $C_j$  ( $j \in J$ ) be sets. Furthermore, suppose that we are given mappings  $r: A \rightarrow B$ ,  $r_i: A \rightarrow B_i$  ( $i \in I$ ),  $g_j: B \rightarrow C_j$  ( $j \in J$ ), and  $f: \prod_{j \in J} C_j \rightarrow D$ .*

(a) *One has  $(g_j)_{j \in J} \circ r = (g_j \circ r)_{j \in J}$ .*

(b) *If  $B = \prod_{i \in I} B_i$ , then  $(g_j)_{j \in J} \circ (r_i)_{i \in I} = (g_j \circ (r_i)_{i \in I})_{j \in J}$ , and thus*

$$(f \circ (g_j)_{j \in J}) \circ (r_i)_{i \in I} = f \circ (g_j \circ (r_i)_{i \in I})_{j \in J}.$$

(c) *If  $B_i = C_j = D = X$  for  $i \in I$  and  $j \in J$ ,  $A = X^k$  and  $I = m$  and  $J = n$ , then we have*

$$\begin{aligned} (f \circ (g_0, \dots, g_{n-1})) \circ (r_0, \dots, r_{m-1}) \\ = f \circ (g_0 \circ (r_0, \dots, r_{m-1}), \dots, g_{n-1} \circ (r_0, \dots, r_{m-1})), \end{aligned}$$

*the superassociativity law for finitary operations on  $X$ .*

As in our modelling natural numbers are sets, we consequently interpret tuples as maps, too: if  $B = n \in \mathbb{N}$  is a natural number, then  $A^B = A^n$  is the set of all  $n$ -tuples  $x = (x(i))_{i < n}$ . We shall often write  $x_i$  for the entry  $x(i)$  ( $i \in n$ ), and, whenever convenient, we shall also refer to the entries of tuples by different indexing, e.g.  $x = (x_1, \dots, x_n)$ . Note that the sole element of  $A^0 = A^\emptyset$  is the empty mapping (tuple), whose graph is the empty relation. It will consistently be denoted by  $\emptyset$ . As tuples are functions we may compose them with other functions: for instance, if  $x \in A^n$  and  $\alpha: m \rightarrow n$ , ( $m, n \in \mathbb{N}$ ), then  $x \circ \alpha$  is the tuple in  $A^m$  whose entries are  $x_{\alpha(i)}$  ( $i \in m$ ). Similarly, if  $g: A \rightarrow B$ , then  $g \circ x = (g(x_i))_{i \in n}$  is an element of  $B^n$ .

Any mapping  $f \in A^{A^n}$  ( $n \in \mathbb{N}$ ) is called an  $n$ -ary operation on  $A$ , and the number  $n$  is referred to as its *arity*, denoted by  $\text{ar}(f)$ . The set of all *finitary operations* on  $A$  is  $O_A := \biguplus_{k \in \mathbb{N}} A^{A^k}$ . Note that we explicitly include nullary operations here, which is slightly uncommon in standard clone theory. For a set of operations  $F \subseteq O_A$  we denote its  $n$ -ary part by  $F^{(n)} := F \cap A^{A^n}$ . We extend this notation to operators yielding subsets of operations: if  $\text{OP}: S \rightarrow \mathfrak{P}(O_A)$  is an operator on a set  $S$ , then we define  $\text{OP}^{(n)}: S \rightarrow \mathfrak{P}(O_A^{(n)})$  by the restriction  $\text{OP}^{(n)}(s) := (\text{OP}(s))^{(n)}$  for  $s \in S$ . Based on this, we put moreover  $\text{OP}^{(n_1, \dots, n_k)}(s) := \biguplus_{i=1}^k \text{OP}^{(n_i)}(s)$  for  $s \in S$  and a finite list of arities  $n_1, \dots, n_k$ ,  $k > 0$ . We also abbreviate  $\text{OP}^{(0, \dots, n)}$  as  $\text{OP}^{(\leq n)}$ , and for  $s \in S$  we let  $\text{OP}^{(>0)}(s) := \text{OP}(s) \setminus \text{OP}^{(0)}(s)$ .

The projection operations belonging to the finite CARTESIAN powers of the carrier set play a special role. For  $n \in \mathbb{N}$  and  $i \in n$ , we denote by  $e_i^{(n)} \in O_A^{(n)}$  the  $n$ -ary projection on the  $i$ -th coordinate. Evidently, there do not exist any nullary projections. Therefore, the set of all projections on  $A$ , denoted by  $J_A$ , equals  $\bigcup_{n \in \mathbb{N}_+} \{e_i^{(n)} \mid 0 \leq i < n\}$ . For the identity operation  $e_0^{(1)}$  we occasionally also use the notation  $\text{id}_A$ .

Knowing about re-indexing tuples, we can recollect the notion of *polymer*. If  $m, n \in \mathbb{N}$  are arities,  $\alpha: n \rightarrow m$  is any indexing map and  $f \in O_A^{(n)}$ , then

$\delta_\alpha(f)$  is the operation in  $O_A^{(m)}$  given by  $\delta_\alpha(f)(x) := f(x \circ \alpha)$  for  $x \in A^m$ . Any operation derived from  $f \in O_A^{(n)}$  by some map  $\alpha: n \rightarrow m$ ,  $m \in \mathbb{N}$ , is said to be a *polymer* of  $f$ . Clearly any polymer of  $f$  can be obtained by composition with a suitable tupling of projections:  $\delta_\alpha(f) = f \circ \left(e_{\alpha(i)}^{(m)}\right)_{i \in n}$ .

Besides operations we shall also need *relations*: for  $m \in \mathbb{N}$  any subset  $\varrho \subseteq A^m$  of  $m$ -tuples is an *m-ary relation* on  $A$ . Thus  $\mathfrak{P}(A^m)$  is the set of all *m-ary relations*, and, again allowing arity equal to null, the set of all *finitary relations* is defined by  $R_A := \bigcup_{\ell \in \mathbb{N}} \mathfrak{P}(A^\ell)$ . If  $Q \subseteq R_A$ , we use  $Q^{(m)} := Q \cap \mathfrak{P}(A^m)$  to denote its *m-ary part*. Moreover, if  $OP: S \rightarrow \mathfrak{P}(R_A)$  is an operator on a set  $S$ , we put  $OP^{(m)}: S \rightarrow \mathfrak{P}(R_A^{(m)})$ , mapping  $s \in S$  to  $OP^{(m)}(s) := (OP(s))^{(m)}$ . Similarly as for operations, for  $s \in S$  we define  $OP^{(\leq m)}(s) := \bigcup_{k=0}^m OP^{(k)}(s)$ ,  $OP^{(\geq m)}(s) := \bigcup_{k \in \mathbb{N}, k \geq m} OP^{(k)}(s)$ , and we let  $OP^{(> m-1)} := OP^{(\geq m)}$ .

A *relation pair* of arity  $m \in \mathbb{N}$  ([16, p. 15] or [18, p. 11]) is any pair  $(\varrho, \varrho')$ , where  $\varrho, \varrho' \in R_A^{(m)}$  and  $\varrho' \subseteq \varrho$ . We collect all *m-ary relation pairs* in the set  $Rp_A^{(m)}$ ; the disjoint union (the importance of this technical aspect is discussed on page 8)  $Rp_A := \bigsqcup_{\ell \in \mathbb{N}} Rp_A^{(\ell)}$  denotes the set of all *finitary relation pairs*. As before, we abbreviate *m-ary parts* as  $Q^{(m)} := Rp_A^{(m)} \cap Q$  for any  $Q \subseteq Rp_A$  and define operator restrictions  $OP^{(m)}: S \rightarrow \mathfrak{P}(Rp_A^{(m)})$  by mapping  $s \in S$  to  $OP^{(m)}(s) := (OP(s))^{(m)}$  for any  $OP: S \rightarrow \mathfrak{P}(Rp_A)$ . Further we put  $OP^{(\leq m)}(s) := \bigsqcup_{k=0}^m OP^{(k)}(s)$  for  $s \in S$ .

There is a natural order relation on  $Rp_A^{(m)}$  for each  $m \in \mathbb{N}$ , which is given by set inclusion in both components. That is, we write  $(\sigma, \sigma') \leq (\varrho, \varrho')$  for  $(\varrho, \varrho'), (\sigma, \sigma') \in Rp_A^{(m)}$  if and only if  $\sigma \subseteq \varrho$  and  $\sigma' \subseteq \varrho'$ . Moreover, we shall need the quasiorder on  $Rp_A^{(m)}$ ,  $m \in \mathbb{N}$ , that is specified by just ordering the first components:  $(\sigma, \sigma') \preceq (\varrho, \varrho')$  holds by definition if and only if  $\sigma \subseteq \varrho$ .

We say that a relation pair  $(\sigma, \sigma') \in Rp_A$  is a *relaxation* of some other pair  $(\varrho, \varrho') \in Rp_A$  (cf. [12, p. 153]) if  $\varrho' \subseteq \sigma'$  and  $\sigma \subseteq \varrho$ . A collection  $Q \subseteq Rp_A$  is *closed w.r.t. relaxations* if with each pair  $(\varrho, \varrho') \in Q$  it also contains any of its relaxations, i.e. if  $\rightarrow Q^+ := \{(\sigma, \sigma') \in Rp_A \mid \exists (\varrho, \varrho') \in Q: \varrho' \subseteq \sigma' \subseteq \sigma \subseteq \varrho\}$  is a subset of (equal to)  $Q$ . Since set inclusion is transitive, the collection  $\rightarrow Q^+$  is the least subset of  $Rp_A$  (w.r.t.  $\subseteq$ ) that contains  $Q$  and is closed w.r.t. relaxations. We call  $\rightarrow Q^+$  the *closure* of  $Q$  w.r.t. *relaxation*. In [18, Definition 1, p. 16] the closure w.r.t. relaxation has been handled by so-called multioperations  $d_v$  and  $d_h$ .

**2.2. The Galois correspondence Polp-Invp.** Here we recall the GALOIS connection Polp-Invp as defined in [16, p. 15] and [18, p. 11]. The formulation is identical except for extending the scope by allowing nullary operations and relations.

**Definition 2.2.** For an  $n$ -ary operation  $f \in O_A^{(n)}$  ( $n \in \mathbb{N}$ ) and an  $m$ -ary relation pair  $(\varrho, \varrho') \in Rp_A^{(m)}$  ( $m \in \mathbb{N}$ ) on a set  $A$ , we say that  $f$  *preserves*  $(\varrho, \varrho')$  and write  $f \triangleright (\varrho, \varrho')$  if the following equivalent conditions hold:

- (i) For every tuple  $\mathbf{r} \in \varrho^n$ , the composition of  $f$  with the tupling  $(\mathbf{r})$  of the tuples in  $\mathbf{r}$  belongs to the smaller relation:  $f \circ (\mathbf{r}) \in \varrho'$ .

- (ii) For every  $(m \times n)$ -matrix  $X \in A^{m \times n}$  the columns  $X_{-,j}$  ( $j \in n$ ) of which are tuples in  $\varrho$ , the tuple  $(f(X_{i,-}))_{i \in m}$  obtained by row-wise application of  $f$  to  $X$  yields a tuple of  $\varrho'$ .

Note in this respect that for any tuple  $\mathbf{r} = (r_j)_{0 \leq j < n} \in (A^m)^n$  where for  $0 \leq j < n$  each tuple is given as  $r_j = (r_{ij})_{0 \leq i < m}$ , the definition of tupling precisely yields that  $f \circ (\mathbf{r}) = \left( f \left( (r_{ij})_{0 \leq j < n} \right) \right)_{0 \leq i < m}$ , i.e. the result of applying  $f$  row-wise to the matrix  $(r_{ij})_{(i,j) \in m \times n} \in A^{m \times n}$ .

Note furthermore, that for  $\varrho \in \mathbf{R}_A$  and  $f \in \mathbf{O}_A$  the condition  $f \triangleright (\varrho, \varrho')$  coincides with the usual preservation condition for functions and relations (cf. [3, Definition 2.3] for the framework involving nullary operations).

Based on the preservation condition we introduce a GALOIS correspondence in the usual way: for a set  $F \subseteq \mathbf{O}_A$  we denote by

$$\text{Invp}_A F := \{ (\varrho, \varrho') \in \mathbf{Rp}_A \mid \forall f \in F: f \triangleright (\varrho, \varrho') \}$$

the set of its *invariant relation pairs*, and, dually, for  $Q \subseteq \mathbf{R}_A$ , the set

$$\text{Polp}_A Q := \{ f \in \mathbf{O}_A \mid \forall (\varrho, \varrho') \in Q: f \triangleright (\varrho, \varrho') \}$$

contains all *polymorphisms of relation pairs* in  $Q$ . The pair  $(\text{Polp}_A, \text{Invp}_A)$  forms the GALOIS correspondence  $\text{Polp-Invp}$ .

If we restrict the latter just to relation pairs  $(\varrho, \varrho')$  where  $\varrho = \varrho'$ , then we get the standard GALOIS connection  $\text{Pol-Inv}$ : for  $F \subseteq \mathbf{O}_A$  we have

$$\{ (\varrho, \varrho) \mid \varrho \in \mathbf{R}_A \wedge \forall f \in F: f \triangleright (\varrho, \varrho) \} = \{ (\varrho, \varrho) \mid \varrho \in \text{Inv}_A F \},$$

where  $\text{Inv}_A F = \{ \varrho \in \mathbf{R}_A \mid \forall f \in F: f \triangleright \varrho \}$ ; and for  $Q \subseteq \{ (\varrho, \varrho) \mid \varrho \in \mathbf{R}_A \}$ , letting  $Q' := \{ \varrho \in \mathbf{R}_A \mid (\varrho, \varrho) \in Q \}$ , it is the case that

$$\text{Polp}_A Q = \{ f \in \mathbf{O}_A \mid \forall (\varrho, \varrho) \in Q: f \triangleright (\varrho, \varrho) \} = \text{Pol}_A Q',$$

wherein  $\text{Pol}_A Q' := \{ f \in \mathbf{O}_A \mid \forall \varrho \in Q': f \triangleright \varrho \}$ .

The name *polymorphism* attributed to the functions in  $\text{Pol}_A Q$  for sets of relations  $Q \subseteq \mathbf{R}_A$  comes from the fact that an operation  $f \in \mathbf{O}_A$  belongs to  $\text{Pol}_A Q$  if and only if it is a homomorphism from the power  $\underline{\mathbf{A}}^{\text{ar}(f)}$  into the relational structure  $\underline{\mathbf{A}} = \langle A; (\varrho)_{\varrho \in Q} \rangle$ . This characterisation can be generalised in the following way.

**Lemma 2.3.** *For  $Q \subseteq \mathbf{Rp}_A$  and any arity  $n \in \mathbb{N}$  an operation  $f \in \mathbf{O}_A^{(n)}$  satisfies  $f \in \text{Polp}_A Q$  if and only if  $f: \langle A; (\varrho)_{(\varrho, \varrho') \in Q} \rangle^n \longrightarrow \langle A; (\varrho')_{(\varrho, \varrho') \in Q} \rangle$  is a homomorphism of relational structures.*

The proof is a straightforward rewriting of the definitions and is therefore omitted.

It is an evident consequence of the definition of preservation that sets of the form  $\text{Invp}_A F$ ,  $F \subseteq \mathbf{O}_A$ , are closed w.r.t. relaxation (cf. [18, Lemma 8, p. 16]).

**Lemma 2.4.** *For  $F \subseteq \mathbf{O}_A$  we have  $\text{Invp}_A F = \neg \text{Invp}_A F^\leftarrow$ .*

*Proof.* Although the statement easily follows from the definition, we present here another argument, based on the local closure  $\text{LOC}_A$  of sets of relation pairs (see Definition 2.8). For any  $Q \subseteq \mathbf{Rp}_A$  we have  $Q \subseteq \neg Q^\leftarrow$ , and we shall prove  $\neg Q^\leftarrow \subseteq \text{LOC}_A Q$  in Corollary 2.10. Hence, we get the inclusions

$\text{Invp}_A F \subseteq \rightarrow \text{Invp}_A F^\leftarrow \subseteq \text{LOC}_A \text{Invp}_A F$ , and we shall see in Corollary 4.10 that  $\text{LOC}_A \text{Invp}_A F = \text{Invp}_A F$ .  $\square$

The following result provides a simple reformulation of the previous lemma.

**Corollary 2.5.** *For every  $Q \subseteq \text{Rp}_A$  we have  $\text{Polp}_A Q = \text{Polp}_A \rightarrow Q^\leftarrow$ .*

*Proof.* By Lemma 2.4,  $Q \subseteq \rightarrow Q^\leftarrow \subseteq \rightarrow \text{Invp}_A \text{Polp}_A Q^\leftarrow = \text{Invp}_A \text{Polp}_A Q$  for  $Q \subseteq \text{Rp}_A$ , so  $\text{Polp}_A Q \supseteq \text{Polp}_A \rightarrow Q^\leftarrow \supseteq \text{Polp}_A \text{Invp}_A \text{Polp}_A Q = \text{Polp}_A Q$ .  $\square$

Some relation pairs are preserved by no operation. If they are part of a set  $Q \subseteq \text{Rp}_A$  or, more generally, part of  $\text{Invp}_A \text{Polp}_A Q$ , then  $\text{Polp}_A Q$  is forced to be empty. The next lemma characterises when this happens (cf. [16, p. 15] and [18, p. 12]).

**Lemma 2.6.** *For  $Q \subseteq \text{Rp}_A$  we have  $\text{Polp}_A Q = \emptyset$  if and only if  $\text{Invp}_A \text{Polp}_A Q$  contains a relation pair of the form  $(\varrho, \emptyset)$  with  $\varrho \neq \emptyset$ , which happens precisely if  $(A^0, \emptyset) \in \text{Invp}_A \text{Polp}_A Q$ .*

*Proof.* If  $\varrho \in \text{Rp}_A$  is non-empty, then also  $\varrho^n \neq \emptyset$  for any possible  $n \in \mathbb{N}$ . Therefore, the condition in Definition 2.2(i) with  $\varrho' = \emptyset$  is not satisfiable for any function  $f \in O_A$ . Hence,  $\text{Polp}_A Q = \text{Polp}_A \text{Invp}_A \text{Polp}_A Q = \emptyset$ , whenever  $(\varrho, \emptyset) \in \text{Invp}_A \text{Polp}_A Q$ .

Conversely, if  $\text{Polp}_A Q = \emptyset$ , then  $\text{Invp}_A \text{Polp}_A Q = \text{Invp}_A \emptyset = \text{Rp}_A$ , which clearly contains the relation pair  $(A^0, \emptyset)$ . The nullary relation  $A^0$  is never empty, even for  $A = \emptyset$ , so the exhibited example is of the right form.  $\square$

**Remark 2.7.** The previous lemma demonstrates the necessity to include nullary relations in the framework, caused by our wish not to impose any restriction on the carrier set  $A$ . Namely, for  $A = \emptyset$ , we have  $A^m = \emptyset$  for all  $m \in \mathbb{N}_+$ , and thus  $\text{R}_A^{(m)} = \mathfrak{P}(A^m) = \{\emptyset\}$ . Hence,  $\text{Rp}_A = \{(A^0, \emptyset)\} \uplus \biguplus_{m \in \mathbb{N}_+} \{(\emptyset, \emptyset)\}$ , which allows us to distinguish between  $\text{Polp}_A \text{Rp}_A = \text{Polp}_A \{(A^0, \emptyset)\} = \emptyset$  and  $\text{Polp}_A \{(\emptyset, \emptyset)\} = \text{Polp}_A \emptyset = O_A$ . Both sets are evidently semiclones (subalgebras of the iterative POST algebra), on any carrier set  $A$ , so, in view of our overall objective, it is more than desirable to be able to model them with our GALOIS correspondence. Restricting to relations of positive arity, this would clearly be impossible for  $A = \emptyset$ .

**2.3. Local closure operators for functions and relation pairs.** For the GALOIS connection  $\text{Pol-Inv}$  in the case of infinite carrier sets, there exist examples  $F \subseteq O_A$  where the inclusion  $\langle F \rangle_{O_A} \subseteq \text{Pol}_A \text{Inv}_A F$  is proper. Hence, in order to characterise the GALOIS closure, an additional local closure operator is needed. A similar situation arises with  $\text{Polp-Invp}$ : for operations we can indeed reuse the same local closure operators as known from  $\text{Pol-Inv}$ . For the side of relation pairs, we have to introduce a new variant of local closure.

In fact, in order to characterise GALOIS closures of sets of at most  $s$ -ary operations / relations ( $s \in \mathbb{N}$ ) we define more specific variants of  $s$ -local closure operators. Note that apart from extending the scope of the definition to  $s = 0$  and nullary operations, the operators  $s\text{-Loc}_A$  and  $\text{Loc}_A$  we define coincide with those from [29, 1.9, p. 15] (see also [28, 1.5, p. 255 et seq.]).

**Definition 2.8.** For  $s \in \mathbb{N}$ ,  $F \subseteq O_A$  and  $Q \subseteq \text{Rp}_A$  we set

$$s\text{-Loc}_A F := \biguplus_{n \in \mathbb{N}} \left\{ g \in O_A^{(n)} \mid \forall B \subseteq A^n, |B| \leq s \exists f \in F^{(n)} : g|_B = f|_B \right\},$$



$$\begin{aligned}
\text{Loc}_A F &:= \bigcap_{s \in \mathbb{N}} s\text{-Loc}_A F, \\
s\text{-LOC}_A Q &:= \bigsqcup_{m \in \mathbb{N}} \left\{ (\sigma, \sigma') \in \text{Rp}_A^{(m)} \mid \begin{array}{l} \forall B \subseteq \sigma, |B| \leq s \exists (\varrho, \varrho') \in Q^{(m)} : \\ B \subseteq \varrho \wedge \varrho' \subseteq \sigma' \end{array} \right\}, \\
\text{LOC}_A Q &:= \bigcap_{s \in \mathbb{N}} s\text{-LOC}_A Q,
\end{aligned}$$

and call these *s-local* and *local closure operators*, respectively.

It is easy to check that  $s\text{-Loc}_A$ ,  $\text{Loc}_A$ ,  $s\text{-LOC}_A$  and  $\text{LOC}_A$  are indeed closure operators on the sets of finitary operations and relation pairs, respectively. Likewise, it is not hard to see that for every  $s, n \in \mathbb{N}$  we have  $s\text{-Loc}_A^{(n)} F = s\text{-Loc}_A (F^{(n)})$  and  $\text{Loc}_A^{(n)} F = \text{Loc}_A (F^{(n)})$  for  $F \subseteq \text{O}_A$ , and similarly we have  $s\text{-LOC}_A^{(n)} Q = s\text{-LOC}_A (Q^{(n)})$  and  $\text{LOC}_A^{(n)} Q = \text{LOC}_A (Q^{(n)})$  for any set  $Q \subseteq \text{Rp}_A$ . To make a technical remark: if we had not insisted on using the disjoint union for the definition of  $\text{Rp}_A$ , then for any  $n \in \mathbb{N}$  we would have  $0\text{-LOC}_A (Q^{(n)}) = \text{Rp}_A$  whenever  $(\emptyset, \emptyset) \in Q$  (as in this case  $(\emptyset, \emptyset) \in Q^{(n)(m)}$  were true for all  $m \in \mathbb{N}$ ), and this would obviously violate the equality mentioned above:  $\text{Rp}_A = 0\text{-LOC}_A (Q^{(n)}) \not\subseteq 0\text{-LOC}_A^{(n)} Q \subseteq \text{Rp}_A^{(n)}$ .

Moreover, it follows directly from the definition that  $t\text{-Loc}_A F \subseteq s\text{-Loc}_A F$  and  $t\text{-LOC}_A Q \subseteq s\text{-LOC}_A Q$  hold for all  $F \subseteq \text{O}_A$  and  $Q \subseteq \text{Rp}_A$  whenever  $s \leq t$ ,  $s, t \in \mathbb{N}$ . Therefore, for  $F \subseteq \text{O}_A$ ,  $Q \subseteq \text{Rp}_A$  and  $s \in \mathbb{N}$  we have the inclusions

$$\begin{aligned}
0\text{-Loc}_A F &\supseteq \cdots \supseteq s\text{-Loc}_A F \supseteq (s+1)\text{-Loc}_A F \supseteq \cdots \supseteq \text{Loc}_A F \supseteq F, \\
0\text{-LOC}_A Q &\supseteq \cdots \supseteq s\text{-LOC}_A Q \supseteq (s+1)\text{-LOC}_A Q \supseteq \cdots \supseteq \text{LOC}_A Q \supseteq Q.
\end{aligned}$$

It follows from these relations that

$$s\text{-Loc}_A t\text{-Loc}_A F = (\min \{s, t\})\text{-Loc}_A F$$

holds for all  $F \subseteq \text{O}_A$  and

$$s\text{-LOC}_A t\text{-LOC}_A Q = (\min \{s, t\})\text{-LOC}_A Q$$

for all  $Q \subseteq \text{Rp}_A$  and any  $s, t \in \mathbb{N} \cup \{\infty\}$  (cp. [29, Proposition 1.10, p. 16]), where we have temporarily put  $\infty\text{-Loc}_A := \text{Loc}_A$  and  $\infty\text{-LOC}_A := \text{LOC}_A$ .

Note that our definition of *s-local* closure of relation pairs for  $s \in \mathbb{N}_+$  entails the corresponding one for relations given in [29, 1.9, p. 16] in the following way: for  $Q' \subseteq \text{R}_A \setminus \text{R}_A^{(0)}$  put  $Q := \bigsqcup_{m \in \mathbb{N}_+} \left\{ (\varrho, \varrho) \mid \varrho \in Q'^{(m)} \right\}$ . Then given  $s > 0$ , one can check that  $s\text{-LOC}_A Q = \bigsqcup_{m \in \mathbb{N}_+} \left\{ (\sigma, \sigma) \mid \sigma \in s\text{-LOC}_A^{(m)} Q' \right\}$  (see Lemma 6.16 for further details), in which  $s\text{-LOC}_A Q'$  denotes the set  $\{\sigma \in \text{R}_A \mid \forall B \subseteq \sigma, |B| \leq s \exists \varrho \in Q' : B \subseteq \varrho \subseteq \sigma\}$ . Hence one may reconstruct  $s\text{-LOC}_A Q'$  as  $\{\sigma \in \text{R}_A \mid (\sigma, \sigma) \in s\text{-LOC}_A Q\}$ . The local closure  $\text{LOC}_A$  of sets of non-nullary relations can be handled in a similar way.

Furthermore, the following characterisation is also simple to verify.

**Lemma 2.9.** *For any set  $A$ , collections  $F \subseteq \text{O}_A$  and  $Q \subseteq \text{Rp}_A$  we have*

$$\text{Loc}_A F = \bigsqcup_{n \in \mathbb{N}} \left\{ g \in \text{O}_A^{(n)} \mid \forall B \subseteq A^n, |B| < \aleph_0 \exists f \in F^{(n)} : g|_B = f|_B \right\},$$



$$\text{LOC}_A Q = \biguplus_{m \in \mathbb{N}} \left\{ (\sigma, \sigma') \in \text{Rp}_A^{(m)} \mid \begin{array}{l} \forall B \subseteq \sigma, |B| < \aleph_0 \exists (\varrho, \varrho') \in Q^{(m)} : \\ B \subseteq \varrho \wedge \varrho' \subseteq \sigma' \end{array} \right\}.$$

From this result it easily follows that closure w.r.t. relaxation is just a special case of the local closure of relation pairs.

**Corollary 2.10.** *For  $Q \subseteq \text{Rp}_A$  we have  $\rightarrow Q^\leftarrow \subseteq \text{LOC}_A Q$ .*

*Proof.* Let  $(\sigma, \sigma')$  be a relaxation of some pair  $(\varrho, \varrho') \in Q^{(m)}$  for some  $m \in \mathbb{N}$ , i.e.  $\varrho' \subseteq \sigma' \subseteq \sigma \subseteq \varrho$ . For any finite subset  $B \subseteq \sigma$  we evidently have  $B \subseteq \sigma \subseteq \varrho$  and  $\varrho' \subseteq \sigma'$  for the relation pair  $(\varrho, \varrho') \in Q^{(m)}$ . Thus, according to Lemma 2.9, it is the case that  $(\sigma, \sigma') \in \text{LOC}_A Q$ .  $\square$

The following consequence is now evident.

**Corollary 2.11.** *Let  $Q \subseteq \text{Rp}_A$  be locally closed (or even  $s$ -locally closed for some  $s \in \mathbb{N}$ ), then it is closed w.r.t. relaxation.*

For relations of fixed arity and finite base sets, there is even a much stronger connection between relaxation and  $s$ -local closure:

**Lemma 2.12.** *For all finite carrier sets  $A$  of cardinality  $k := |A| < \aleph_0$  and any  $m \in \mathbb{N}$ , we have  $\rightarrow Q^\leftarrow = \text{LOC}_A Q = k^m\text{-LOC}_A Q$  for all  $Q \subseteq \text{Rp}_A^{(m)}$ .*

*Proof.* The inclusions  $\rightarrow Q^\leftarrow \subseteq \text{LOC}_A Q \subseteq k^m\text{-LOC}_A Q$  hold in general (cf. Corollary 2.10). Conversely, for a set  $Q \subseteq \text{Rp}_A^{(m)}$  of  $m$ -ary pairs, let us consider any  $(\sigma, \sigma') \in k^m\text{-LOC}_A Q = k^m\text{-LOC}_A (Q^{(m)}) = k^m\text{-LOC}_A^{(m)} Q$ . As  $\sigma \in R_A^{(m)}$ , we have  $|\sigma| \leq |A^m| = k^m$ . Hence, taking  $B := \sigma$  as a subset of  $\sigma$  having at most  $k^m$  elements, by definition of  $k^m\text{-LOC}_A$ , we get a pair  $(\varrho, \varrho') \in Q^{(m)}$  such that  $\sigma = B \subseteq \varrho$  and  $\varrho' \subseteq \sigma'$ . Therefore,  $(\sigma, \sigma') \in \rightarrow Q^\leftarrow$ .  $\square$

In particular, in case of finite carrier sets, the inclusion in Corollary 2.10 is always an equality.

**Corollary 2.13.** *For finite  $A$  we have  $\rightarrow Q^\leftarrow = \text{LOC}_A Q$  for all  $Q \subseteq \text{Rp}_A$ ; in particular, a subset  $Q \subseteq \text{Rp}_A$  is locally closed if and only if it is closed w.r.t. relaxation.*

*Proof.* The set  $\text{LOC}_A Q = \biguplus_{m \in \mathbb{N}} \text{LOC}_A^{(m)} Q = \biguplus_{m \in \mathbb{N}} \text{LOC}_A (Q^{(m)})$  is equal to  $\biguplus_{m \in \mathbb{N}} \rightarrow Q^{(m)\leftarrow} = \rightarrow Q^\leftarrow$  upon application of Lemma 2.12.  $\square$

The following closure property will become important regarding the characterisation of the closure operator  $\text{Invp}_A \text{Polp}_A^{(\leq s)}$  in Section 5. For  $s \in \mathbb{N}$  a collection  $\mathcal{T} \subseteq \mathfrak{P}(S)$  of subsets of a set  $S$  is called  *$s$ -directed* if and only if for all  $t \leq s$ , all  $(X_i)_{i \in t} \in \mathcal{T}^t$  and every  $\mathbf{r} = (r_i)_{i \in t} \in \prod_{i \in t} X_i$  there is a set  $Z \in \mathcal{T}$  such that  $\text{im } \mathbf{r} = \{r_i \mid i \in t\} \subseteq Z$ . Clearly, this condition is equivalent to  $\mathcal{T}$  being non-empty and that for all  $(X_i)_{i \in s} \in \mathcal{T}^s$  and  $\mathbf{r} \in \prod_{i \in s} X_i$  there exists  $Z \in \mathcal{T}$  fulfilling  $\text{im } \mathbf{r} \subseteq Z$ . We say that a set  $Q \subseteq \text{Rp}_A^{(m)}$  of  $m$ -ary relation pairs is  *$s$ -directed* if and only if  $\{\varrho \mid (\varrho, \varrho') \in Q\} \subseteq \mathfrak{P}(A^m)$  is  $s$ -directed in the sense above. We prove now that sets of the form  $s\text{-LOC}_A Q$ , where  $Q \subseteq \text{Rp}_A$ , are closed w.r.t. unions of  $s$ -directed systems of relation pairs of the same arity.

**Lemma 2.14.** *If  $s, m \in \mathbb{N}$ ,  $Q \subseteq \text{Rp}_A$ , and  $\mathcal{T} \subseteq s\text{-LOC}_A^{(m)} Q$  is  $s$ -directed, then we have  $\bigcup \mathcal{T} := \left( \bigcup_{(\mu, \mu') \in \mathcal{T}} \mu, \bigcup_{(\mu, \mu') \in \mathcal{T}} \mu' \right) \in s\text{-LOC}_A^{(m)} Q$ .*

*Proof.* Clearly, we have  $\sigma' := \bigcup_{(\mu, \mu') \in \mathcal{T}} \mu' \subseteq \bigcup_{(\mu, \mu') \in \mathcal{T}} \mu =: \sigma$ , so the union  $(\sigma, \sigma')$  is a well-defined relation pair in  $\text{Rp}_A^{(m)}$ . In order to prove that it belongs to  $s\text{-LOC}_A Q$ , we consider any subset  $B = \{b_i \mid i \in t\} \subseteq \sigma$  such that  $t := |B| \leq s$ . By definition of  $\sigma$ , for each  $i \in t$  there exists a pair  $(\mu_i, \mu'_i) \in \mathcal{T}$  such that  $b_i \in \mu_i$ . By  $s$ -directedness of  $\mathcal{T}$  there exists some  $(\mu, \mu') \in \mathcal{T}$  such that  $B = \{b_i \mid i \in t\} \subseteq \mu$ . For  $(\mu, \mu') \in \mathcal{T} \subseteq s\text{-LOC}_A^{(m)} Q$  and  $B \subseteq \mu$  has at most  $s$  elements, there must exist some pair  $(\varrho, \varrho') \in Q^{(m)}$  such that  $B \subseteq \varrho$  and  $\varrho' \subseteq \mu' \subseteq \sigma'$ . This shows that  $(\sigma, \sigma') \in s\text{-LOC}_A^{(m)} Q$ .  $\square$

For  $m \in \mathbb{N}$  and we say that a set  $\mathcal{T} \subseteq \text{Rp}_A^{(m)}$  is  $\aleph_0$ -directed if it is  $s$ -directed for all  $s \in \mathbb{N}$ . This means we require the condition presented before Lemma 2.14 to hold for any finite sequence of relations and tuples.

We call a set  $\mathcal{T} \subseteq \text{Rp}_A^{(m)}$  directed if  $\mathcal{T} \neq \emptyset$  and for all  $(\varrho_1, \varrho'_1), (\varrho_2, \varrho'_2) \in \mathcal{T}$  there exists some  $(\varrho, \varrho') \in \mathcal{T}$  such that  $\varrho_1 \cup \varrho_2 \subseteq \varrho$ . This is equivalent to saying that for any finite subset  $\mathcal{F} \subseteq \mathcal{T}$  there is an pair  $(\varrho, \varrho') \in \mathcal{T}$  such that  $\bigcup_{(\mu, \mu') \in \mathcal{F}} \mu \subseteq \varrho$ , wherefore directedness clearly implies  $\aleph_0$ -directedness.

As a consequence of this implication we get that locally closed sets of relation pairs are closed under directed unions of sets of pairs of identical arity.

**Corollary 2.15.** *For all  $m \in \mathbb{N}$ ,  $Q \subseteq \text{Rp}_A$  and every  $\aleph_0$ -directed collection  $\mathcal{T} \subseteq \text{LOC}_A^{(m)} Q$ , we have  $\bigcup \mathcal{T} := \left( \bigcup_{(\mu, \mu') \in \mathcal{T}} \mu, \bigcup_{(\mu, \mu') \in \mathcal{T}} \mu' \right) \in \text{LOC}_A^{(m)} Q$ . In particular this is true whenever  $\mathcal{T} \subseteq \text{LOC}_A^{(m)} Q$  is directed.*

Under additional assumptions on the set of relation pairs  $Q$  we shall extend Lemma 2.14 and Corollary 2.15 to characterisations of local and  $s$ -local closedness. We conclude this subsection with remarks on the relationship of our local closure operators to others defined in the more general setting treated in [11].

**Remark 2.16.** The local closure operators (and  $s$ -local closure operators for  $s \in \mathbb{N}_+$ ) defined here cannot directly be derived as special cases of the corresponding closure operators from [11]. As the case of local closures is similar, we shall only argue for  $s$ -local closures. Specialising the framework in the mentioned article for a pair of carrier sets  $(A, B)$  where  $B = A$ , we may apply the  $s$ -local closure  $\mathbf{LO}_s$  described there to any set  $Q \subseteq \text{Rp}_A \setminus \text{Rp}_A^{(0)}$ , then yielding the collection

$$\mathbf{LO}_s(Q) = Q \cup \bigcup_{m \in \mathbb{N}_+} \left\{ (R, S) \in (\mathfrak{P}(A^m))^2 \mid \forall C \subseteq R, |C| \leq s \forall A^m \supseteq T \supseteq S: (C, T) \in Q \right\}.$$

As this set contains pairs  $(R, S)$  that are not relation pairs, i.e. failing the condition  $R \supseteq S$ , the canonical modification would be to simply intersect the result with  $\text{Rp}_A$ , leading to

$$\mathbf{LO}_s(Q) \cap \text{Rp}_A = Q \cup \bigcup_{m \in \mathbb{N}_+} \left\{ (R, S) \in \text{Rp}_A^{(m)} \mid \forall C \subseteq R, |C| \leq s \forall A^m \supseteq T \supseteq S: (C, T) \in Q \right\}.$$

This set equals  $Q$  on any set  $A$  (in fact, the second part of the union is empty, whenever  $A \neq \emptyset$ , as for  $C = \emptyset$  and  $T = A^m \neq \emptyset$  the condition  $(C, T) \in Q$  is never satisfied). So the original definition of  $\mathbf{LO}_s$  (or its canonical modification) is not helpful at all in our setting.

Suppose, in the union over  $m \in \mathbb{N}_+$ , we change the condition describing when a relation pair  $(R, S)$  is added to the  $s$ -local closure of  $Q$  as follows: among all relational constraints  $(C, T)$  relaxing  $(R, S)$  and verifying  $|C| \leq s$  only those are required to be in  $Q$  that *are indeed relation pairs*. Then we get

$$Q \cup \bigcup_{m \in \mathbb{N}_+} \left\{ (R, S) \in \text{Rp}_A^{(m)} \mid \forall C \subseteq R, |C| \leq s \forall C \supseteq T \supseteq S: (C, T) \in Q \right\}.$$

This set still differs from  $s\text{-LOC}_A Q$  as defined above. For instance for any  $s \in \mathbb{N}_+$  and  $Q = \emptyset$  we have  $s\text{-LOC}_A Q = \emptyset$ , while the previously displayed collection contains all relation pairs  $(R, S) \in \text{Rp}_A$  where  $|S| > s$ .

We do not see an obvious way how to translate  $\mathbf{LO}_s$  into  $s\text{-LOC}_A$  or vice versa.

### 3. SEMICLONES AND THE FULL ITERATIVE POST ALGEBRA

The following definition is very similar to that of a clone of operations. The only difference is that a clone  $F \subseteq \text{O}_A$  is additionally required to contain the set  $\text{J}_A$  of projections as a subset.

**Definition 3.1.** A (concrete) semicclone (of operations) on a set  $A$  is a subset  $F \subseteq \text{O}_A$  of all finitary operations such that for all  $m, n \in \mathbb{N}$  we have  $f \circ (g_0, \dots, g_{n-1}) \in F$  for each  $f \in F^{(n)}$  and  $(g_0, \dots, g_{n-1}) \in (F \cup \text{J}_A)^{(m)}^n$ .

The closure property stated in Definition 3.1 is formulated in terms of partial composition operations on  $\text{O}_A$  as the functions making up the tupling all have to be of identical arity. However, it is possible to extend these operations in a conservative way to totally defined operations on  $\text{O}_A$  such that semiclones are exactly the subuniverses of a certain universal algebra on the carrier set  $\text{O}_A$ : for each  $n, m \in \mathbb{N}$  and each subset  $I \subseteq n$  and any tuple  $(g_i)_{i \in I} \in \text{J}_A^{(m)}$  of  $m$ -ary projections we define an  $(|n \setminus I| + 1)$ -ary operation on  $\text{O}_A$ , which maps  $(f, (g_i)_{i \in n \setminus I})$  to  $f \circ (g_i)_{i \in n}$  provided that  $f \in \text{O}_A^{(n)}$  and  $g_i \in \text{O}_A^{(m)}$  for all  $i \in n \setminus I$ , and to  $f$  otherwise. If we collect all the finitary operations obtained in this way in a set  $\Phi \subseteq \text{O}_{\text{O}_A}$ , then it becomes clear that  $F \subseteq \text{O}_A$  is a semicclone if and only if it is a subuniverse of the algebra  $\langle \text{O}_A; \Phi \rangle$ .

Hence, the set  $\mathcal{S}_A := \text{Sub}(\langle \text{O}_A; \Phi \rangle)$  of all semiclones on  $A$  bears the structure of a complete algebraic lattice w.r.t. set-inclusion, and is, in particular, a closure system. The corresponding closure operator will be denoted by  $[\ ]_{\text{O}_A}$ .

Evident, trivial examples of semiclones are the empty set of operations and any clone  $F \subseteq \text{O}_A$ . Moreover, we have the following class of examples:

**Lemma 3.2.** For a set  $G \subseteq \text{O}_A^{(1)}$  of unary transformations, abbreviate its generated transformation semigroup by  $S := \langle G \rangle_{\langle \text{O}_A^{(1)}; \circ \rangle}$ . Then we have

$$[G]_{\text{O}_A} = \left\{ f \circ e_i^{(n)} \mid i \in n \wedge n \in \mathbb{N}_+ \wedge f \in S \right\}.$$

*Proof.*  $S$  is obtained from  $G$  by closure w.r.t. composition of unary operations, which is part of the requirement in Definition 3.1. Thus, we have  $S \subseteq [G]_{\text{O}_A}$ ,

but now the closure property clearly yields that  $[G]_{O_A}$  must contain the whole set on the right-hand side as a subset.

Conversely, it is easy to check that the latter collection, first of all, contains  $S$  and therefore  $G$ , and second, actually forms a semiclone. Thus, it must be a superset of the least semiclone containing  $G$ , which is  $[G]_{O_A}$ .  $\square$

**Corollary 3.3.** *The unary parts of semiclones  $\{F^{(1)} \mid F \in \mathcal{S}_A\}$  are precisely all (carrier sets of) transformation semigroups on  $A$ .*

*Proof.* By definition, the restriction  $F^{(1)}$  of any semiclone  $F \in \mathcal{S}_A$  forms a transformation semigroup. The converse inclusion follows from Lemma 3.2 as we have  $[S]_{O_A}^{(1)} = \left\{ f \circ e_0^{(1)} \mid f \in S \right\} = S$  for any transformation semigroup  $S \subseteq O_A$ .  $\square$

As mentioned in the introduction, semiclones are not a new invention. They are just the subuniverses (“closed classes of functions”) of the full iterative POST algebra. In order to see this we need a few definitions.

For  $n \in \mathbb{N}_+$  define  $\alpha_n^\zeta: n \rightarrow n$  by  $\alpha_n^\zeta(i) := i + 1 \pmod{n}$  and  $\alpha_0^\zeta := \text{id}_0$ . Moreover, let  $\alpha_n^\tau: n \rightarrow n$  be the transposition  $(0, 1)$  for  $n \in \mathbb{N}$ ,  $n \geq 2$ , and put  $\alpha_n^\tau := \text{id}_n$  for  $n \in \{0, 1\}$ . We continue by defining  $\alpha_n^\Delta: n \rightarrow n - 1$  via  $\alpha_n^\Delta(i) := \max(0, i - 1)$  for  $n \in \mathbb{N}$ ,  $n \geq 2$ , letting  $\alpha_n^\Delta := \text{id}_n$  for  $n \in \{0, 1\}$  and declaring the map  $\alpha_n^\nabla: n \rightarrow n + 1$  by  $\alpha_n^\nabla(i) := i + 1$  for any  $n \in \mathbb{N}$ .

On this basis we define for  $\omega \in \{\zeta, \tau, \Delta, \nabla\}$  a unary map  $\omega: O_A \rightarrow O_A$  by  $\omega(f) := \delta_{\alpha_{\text{ar}(f)}^\omega}(f)$  for  $f \in O_A$ . Moreover, for  $f, g \in O_A$ ,  $n := \text{ar}(f)$ ,  $m := \text{ar}(g)$  we construct  $f * g \in O_A^{(k)}$  where  $k := \max(0, n + m - 1)$  as follows: if  $n \geq 2$  we put  $f * g := f \circ \left( g \circ \left( e_i^{(k)} \right)_{i \in m}, \left( e_{m+j}^{(k)} \right)_{j \in n-1} \right)$ ; for  $n = 1$ , we define the product  $f * g := f \circ g \circ \left( e_i^{(k)} \right)_{i \in m}$  whenever  $m > 0$ , and  $f * g := f \circ g$  if  $m = 0$ ; for  $n = 0$ , we define  $f * g := f$  in case that  $k = 0$ , and  $f * g := f \circ \left( e_i^{(k)} \right)_{i \in 0}$  otherwise (where  $\left( e_i^{(k)} \right)_{i \in 0}$  by definition is the unique map from  $A^k$  to  $A^0$ ).

In this way, we obtain an algebra  $O_A := \langle O_A; \zeta, \tau, \Delta, \nabla, * \rangle$  of arity type  $(1, 1, 1, 1, 2)$  that we call *full iterative POST algebra*. It is easy to see that  $O_A \setminus O_A^{(0)}$  is a subuniverse, and the corresponding subalgebra is the one that has been introduced under precisely the same name in [26]. The difference in terminology is just of technical nature and shows up because we wish to accommodate all nullary constants in our framework.

The algebra  $O_A$  obviously is less prodigal of its fundamental operations than  $\langle O_A; \Phi \rangle$  introduced above. The following lemma proves that both actually do the same job.

**Lemma 3.4.** *The semiclones on  $A$  are exactly the subuniverses of the full iterative POST algebra:  $\mathcal{S}_A = \text{Sub}(O_A)$ .*

*Proof.* We saw earlier that any polymer can be expressed as a composition with a tupling of projections under which semiclones are closed by definition. Thus any semiclone is closed w.r.t. the unary operations  $\zeta$ ,  $\tau$ ,  $\Delta$  and  $\nabla$ . By construction of  $*$ , it is also closed w.r.t.  $*$ .

It is a tedious, but well-known exercise (known from the proof that clones are exactly the subuniverses of function algebras, which differ from iterative algebras by just adding an additional constant representing a projection) that the converse also holds: for arities  $m, n \in \mathbb{N}$  and operations  $f \in F^{(n)}$  and  $g_0, \dots, g_{n-1} \in \left((F \cup J_A)^{(m)}\right)^n$  any composition  $f \circ (g_0, \dots, g_{n-1})$  can be expressed as the result of a term operation of  $O_A$  applied to  $(f, g_0, \dots, g_{n-1})$ . For  $F \in \text{Sub}(O_A)$  this means that any such composition also has to belong to  $F$ .  $\square$

The following facts on the relationship of semiclones and clones are well-known (see [16, p. 5 et seq.] or [17, p. 8 et seq.], Lemmata 3 and 4, and Satz 1). In this context, we recollect that  $\langle F \rangle_{O_A}$  denotes the least clone containing some set  $F \subseteq O_A$ , i.e. the *clone generated by  $F$* . The symbol  $\mathcal{L}_A$  stands for the set of all clones on  $A$ .

**Lemma 3.5.** *For any set  $F \subseteq O_A$  and any  $0 \leq i < n$ ,  $n \in \mathbb{N}$ , the following assertions are true:*

- (a)  $\left[\left\{e_i^{(n)}\right\}\right]_{O_A} = J_A$ .
- (b)  $\left[F \cup \left\{e_i^{(n)}\right\}\right]_{O_A} = [F]_{O_A} \cup J_A = \langle F \rangle_{O_A}$ .
- (c) If  $F \in \mathcal{S}_A$ , then  $F \cap J_A \in \{\emptyset, J_A\}$ .
- (d)  $\mathcal{L}_A = \{G \in \mathcal{S}_A \mid G \cap J_A \neq \emptyset\}$ .

*Proof.* Fix any set  $F \subseteq O_A$  and any projection  $e_i^{(n)}$ , where  $0 < i \leq n$ ,  $n \in \mathbb{N}_+$ .

- (a) The set of projections is a clone, and hence a semiclon. We only need to check that  $e_i^{(n)}$  generates any other projection. First, we note that  $\text{id}_A = e_i^{(n)} \circ (\text{id}_A, \dots, \text{id}_A)$ , so  $\text{id}_A \in \left[\left\{e_i^{(n)}\right\}\right]_{O_A}$ . Besides, for any  $m \in \mathbb{N}_+$  and  $0 \leq j < m$  we have  $e_j^{(m)} = \text{id}_A \circ e_i^{(m)}$ , whence we obtain the inclusion  $J_A \subseteq [\{\text{id}_A\}]_{O_A} \subseteq \left[\left\{e_i^{(n)}\right\}\right]_{O_A}$ .
- (b) The relation  $[F]_{O_A} \cup J_A \subseteq \left[F \cup \left\{e_i^{(n)}\right\}\right]_{O_A}$  follows from (a), and the inclusion  $\left[F \cup \left\{e_i^{(n)}\right\}\right]_{O_A} \subseteq \langle F \rangle_{O_A}$  holds as each clone is a semiclon. Finally,  $[F]_{O_A} \cup J_A$  contains  $F$ , and it is easy to check that it is indeed a clone. Therefore, it has to contain  $\langle F \rangle_{O_A}$  as a subset.
- (c) If  $F$  is a semiclon and  $F \cap J_A \neq \emptyset$ , then for some arity  $n \in \mathbb{N}$  and some  $0 \leq i < n$  we have  $F = [F]_{O_A} = \left[F \cup \left\{e_i^{(n)}\right\}\right]_{O_A} \supseteq J_A$  by (b). Therefore,  $F \cap J_A = J_A$ .
- (d) The inclusion “ $\subseteq$ ” is trivial. Conversely, any semiclon  $G \in \mathcal{S}_A$  such that  $G \cap J_A \neq \emptyset$  fulfils  $J_A \subseteq G$  by (c). Hence, by (b), one obtains that  $\langle G \rangle_{O_A} = [G]_{O_A} \cup J_A = G \cup J_A = G$ , i.e. that  $G$  is a clone.  $\square$

As a consequence of the previous lemma, we can describe those semiclones whose unary parts yield proper transformation semigroups, i.e. those which are no monoids.

**Corollary 3.6.** *On any set  $A$  we have*

$$\left\{F^{(1)} \mid F \in \mathcal{S}_A \setminus \mathcal{L}_A\right\} = \left\{S \subseteq O_A^{(1)} \setminus \{\text{id}_A\} \mid (S, \circ) \text{ is a semigroup}\right\}.$$

*Proof.* If  $F \in \mathcal{S}_A \setminus \mathcal{L}_A$ , then  $F^{(1)}$  is a carrier set of a transformation semigroup by Corollary 3.3. Since  $F \in \mathcal{S}_A \setminus \mathcal{L}_A$ , we have  $F \cap J_A = \emptyset$  by Lemma 3.5(d), and so  $F^{(1)} \subseteq O_A^{(1)} \setminus \{\text{id}_A\}$ . Conversely, if  $S \subseteq O_A^{(1)} \setminus \{\text{id}_A\}$  is a carrier set of a proper transformation semigroup, then, by Corollary 3.3, there exists some  $F \in \mathcal{S}_A$  such that  $S = F^{(1)}$ . If  $F \in \mathcal{L}_A$ , then we would have  $\text{id}_A \in F^{(1)} = S$ , violating our assumption. Hence,  $F \in \mathcal{S}_A \setminus \mathcal{L}_A$ .  $\square$

The GALOIS correspondence  $\text{Polp} - \text{Invp}$  gives us plenty of examples of semiclones (cf. [18, Lemma 2, p. 12] for the situation without nullary operations).

**Lemma 3.7.** *Any polymorphism set  $\text{Polp}_A Q$  with  $Q \subseteq \text{Rp}_A$  is a semiclone.*

*Proof.* Consider any  $Q \subseteq \text{Rp}_A$  and put  $Q_1 := \{\varrho \in \text{Rp}_A \mid (\varrho, \varrho') \in Q\}$ . Let  $m, n \in \mathbb{N}$ ,  $f \in \text{Polp}_A^{(n)} Q$  and  $(g_0, \dots, g_{n-1}) \in (\text{Pol}_A^{(m)} Q_1)^n$ . We prove that the composition  $h := f \circ (g_0, \dots, g_{n-1})$  belongs to  $\text{Polp}_A Q$ , which demonstrates our lemma as obviously  $J_A \cup \text{Polp}_A Q \subseteq \text{Pol}_A Q_1$ . Indeed,  $h$  preserves any relation pair  $(\varrho, \varrho') \in Q$ : whenever  $\mathbf{r} \in \varrho^m$ , then superassociativity yields

$$h \circ (\mathbf{r}) = (f \circ (g_0, \dots, g_{n-1})) \circ (\mathbf{r}) \stackrel{2.1(c)}{=} f \circ (g_0 \circ (\mathbf{r}), \dots, g_{n-1} \circ (\mathbf{r})).$$

The latter tuple is a member of  $\varrho'$  as  $f \triangleright (\varrho, \varrho')$  and  $(g_0 \circ (\mathbf{r}), \dots, g_{n-1} \circ (\mathbf{r}))$  belongs to  $\varrho^n$  due to  $g_0, \dots, g_{n-1} \in \text{Pol}_A Q_1 \subseteq \text{Pol}_A \{\varrho\}$ .  $\square$

The following facts can be routinely proven using Lemma 3.7.

**Corollary 3.8.** *For any set  $F \subseteq O_A$  we have*

$$[F]_{O_A} \subseteq \text{Polp}_A \text{Invp}_A F \quad \text{and} \quad \text{Invp}_A F = \text{Invp}_A [F]_{O_A}.$$

The next lemma (cf. [18, Lemma 3, p. 13]) clarifies which sets of relation pairs yield proper clones.

**Lemma 3.9.** *For  $Q \subseteq \text{Rp}_A$  a semiclone  $\text{Polp}_A Q$  is a clone if and only if  $\varrho = \varrho'$  holds for all  $(\varrho, \varrho') \in Q$ .*

*Proof.* If  $Q \subseteq \text{Rp}_A$  only consists of identical pairs, then we saw already in Subsection 2.2 that  $\text{Polp}_A Q = \text{Pol}_A Q'$  where  $Q' = \{\varrho \in \text{Rp}_A \mid (\varrho, \varrho) \in Q\}$ . This set always is a clone. On the other hand, if  $\text{Polp}_A Q$  is a clone, then we have  $\text{id}_A \in \text{Polp}_A Q$ , which implies  $\varrho \subseteq \varrho'$  and thus  $\varrho = \varrho'$  for every  $(\varrho, \varrho') \in Q$ .  $\square$

Note that (along with an appropriate generalisation of preservation) the three previous statements remain true if one considers relation pairs of arbitrary, possibly infinite arity. That is to say, pairs  $(R, S)$ , where  $S \subseteq R \subseteq A^K$  for some fixed set  $K$ .

The following two results are in close analogy to Proposition 1.11(a),(b) from [29, p. 17].

**Lemma 3.10.** *For  $s \in \mathbb{N}$  and any set  $Q \subseteq \text{Rp}_A^{(\leq s)} := \bigsqcup_{0 \leq m \leq s} \text{Rp}_A^{(m)}$  of at most  $s$ -ary relation pairs, we have  $s\text{-Loc}_A \text{Polp}_A Q = \text{Polp}_A Q$ .*

*Proof.* Consider  $n \in \mathbb{N}$  and  $g \in s\text{-Loc}_A^{(n)} \text{Polp}_A Q$ . To prove that  $g \in \text{Polp}_A Q$  take  $(\varrho, \varrho') \in Q^{(m)}$  for some  $0 \leq m \leq s$ ; we have to check that  $g \triangleright (\varrho, \varrho')$ . For this consider any  $n$ -tuple  $\mathbf{r} = (r_j)_{0 \leq j < n} \in \varrho^n$  of tuples from  $\varrho$  and define  $B := \left\{ (r_j(i))_{0 \leq j < n} \mid 0 \leq i < m \right\}$ . Evidently,  $B$  is a subset of the domain  $A^n$  of  $g$  and satisfies  $|B| \leq m \leq s$ . Hence, as  $g \in s\text{-Loc}_A \text{Polp}_A Q$ , there exists

some  $f \in \text{Polp}_A^{(n)} Q$  such that  $g|_B = f|_B$ . This implies that  $g \circ (\mathbf{r}) = f \circ (\mathbf{r})$ , and the latter tuple belongs to  $\varrho'$  as  $f \triangleright (\varrho, \varrho') \in Q$ .  $\square$

**Corollary 3.11.** *The equality  $\text{Loc}_A \text{Polp}_A Q = \text{Polp}_A Q$  is satisfied for any  $Q \subseteq \text{Rp}_A$ .*

*Proof.* Consider  $g \in \text{Loc}_A \text{Polp}_A Q$  and  $(\varrho, \varrho') \in Q^{(m)}$ ,  $m \in \mathbb{N}$ . By definition of  $\text{Loc}_A$ , we have  $g \in m\text{-Loc}_A \text{Polp}_A Q \subseteq m\text{-Loc}_A \text{Polp}_A \{(\varrho, \varrho')\}$ , which equals  $\text{Polp}_A \{(\varrho, \varrho')\}$  by Lemma 3.10. As the pair  $(\varrho, \varrho') \in Q$  was arbitrarily chosen, we obtain  $g \in \text{Polp}_A Q$ .  $\square$

#### 4. RELATION PAIR CLONES

In this section we first recollect the so-called *general superposition* of relations ([29, Definition 3.4(R4), p. 27], see also [28, Definition 2.2(ii), p. 258] and [3]), which comes into play when generalising the notion of relational clone from finite carrier sets to arbitrary ones. It is not surprising that it will be important for the generalisation of relation pair algebras as introduced in [16, p. 21] (see also [18, p. 16]) to carrier sets of arbitrary cardinality, as well.

**Definition 4.1.** Let  $A$  be any carrier set, moreover let index sets  $I$  and  $\mu$  (one could in principle restrict to ordinal numbers, but this only makes working with the definition more technical), natural numbers  $m, m_i \in \mathbb{N}$  ( $i \in I$ ), mappings  $(\alpha_i: m_i \rightarrow \mu)_{i \in I}$  and  $\beta: m \rightarrow \mu$ , and relations  $\varrho_i \in \text{R}_A^{(m_i)}$ ,  $i \in I$ , be given. The *general superposition* of these relations w.r.t. the given data is defined to be the  $m$ -ary relation

$$\bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i)_{i \in I} := \{y \in A^m \mid \exists a \in A^\mu: y = a \circ \beta \wedge \forall i \in I: a \circ \alpha_i \in \varrho_i\} \\ = \{a \circ \beta \mid a \in A^\mu \wedge \forall i \in I: a \circ \alpha_i \in \varrho_i\}.$$

We mention in passing that, in general, a *relational clone* can be defined as any set  $Q \subseteq \text{R}_A$  that is closed w.r.t. general superposition. That is, whenever data as in Definition 4.1 is given and all relations  $\varrho_i$ ,  $i \in I$ , belong to  $Q$ , then also  $\bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i)_{i \in I}$  has to be an element of  $Q$  (if nullary relations are disregarded, then one restricts the integers  $m$  and  $(m_i)_{i \in I}$  to positive ones only). Depending on the carrier set  $A$ , one can work out cardinality bounds on the sets  $I$  and  $\mu$  involved in this closure property, but this is not our concern here.

Different specialisations of the general superposition yield operations known from the closure property corresponding to relational clones on finite carrier sets: variable permutation, projection onto arbitrary subsets of coordinates, variable identification, addition of fictitious coordinates, all diagonal relations as nullary constants, and (even arbitrary) intersection of relations of the same arity.

We now straightforwardly extend the general superposition from relations to relation pairs.

**Definition 4.2.** Let  $A$  be any carrier set,  $I, \mu, m, m_i, \alpha_i: m_i \rightarrow \mu$ ,  $i \in I$ , and  $\beta: m \rightarrow \mu$  as in Definition 4.1. For relation pairs  $(\varrho_i, \varrho'_i) \in \text{Rp}_A^{(m_i)}$ ,  $i \in I$ ,



we define their *general superposition* to be

$$\bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i, \varrho'_i)_{i \in I} := \left( \bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i)_{i \in I}, \bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho'_i)_{i \in I} \right).$$

It is easy to see that this definition is well-defined, i.e. that we really have  $\bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i, \varrho'_i)_{i \in I} \in \text{Rp}_A^{(m)}$  in the situation described in Definition 4.2. This allows us to define relation pair clones as such sets of relation pairs that are closed under general superposition.

**Definition 4.3.** We say that for some carrier  $A$  a set  $Q \subseteq \text{Rp}_A$  is a *relation pair clone* if and only if the following condition is satisfied: whenever  $I, \mu, m, m_i, \alpha_i: m_i \rightarrow \mu, i \in I$  and  $\beta: m \rightarrow \mu$  are as in Definition 4.1, and  $(\varrho_i, \varrho'_i) \in Q^{(m_i)}$  are given for  $i \in I$ , then also  $\bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i, \varrho'_i)_{i \in I} \in Q^{(m)}$ .

One can routinely check that for a given carrier set  $A$  the collection of all relation pair clones on  $A$  is a closure system. We denote the corresponding closure operator by  $Q \mapsto [Q]_{\text{Rp}_A}$  for  $Q \subseteq \text{Rp}_A$  and refer to  $[Q]_{\text{Rp}_A}$  as the *relation pair clone generated by  $Q$* .

Note that for finite carrier sets  $A \neq \emptyset$ , and provided that  $(\emptyset, \emptyset) \in Q^{(m)}$  for all  $m \in \mathbb{N}$ , our concept of locally closed relation pair clone, by taking  $Q \setminus \text{Rp}_A^{(0)}$ , subsumes that of subuniverses of the full relation pair algebra defined in [16, p. 21] (see also [18, p. 16]).

There are two issues here: the necessity to add local closure and the requirement that pairs of empty relations have to belong to relation pair algebras in Harnau's sense. We noted above in Corollary 2.13 that for finite carrier sets closure under relaxation coincides with our local closure of relation pairs. Moreover, we shall prove in Corollary 5.14 that the closed sets w.r.t.  $\text{Invp}_A \text{Polp}_A$  are precisely the locally closed relation pair clones, which implies for finite carrier sets that they are exactly those relation pair clones that are closed w.r.t. relaxations. In [16] and [18] this additional closure property (with the goal of characterising the GALOIS closures) has been incorporated into the definition of the full relation pair algebra via multioperations  $d_v$  and  $d_h$ ; however, it has been noted that these operators are of a different nature than the other fundamental operations of relation pair algebras. Comparing to the situation known from clones and relational clones on arbitrary domains (see [29, 28, 33]) and looking from the perspective of infinite carrier sets, which requires local closures anyway, it is justified to modify Harnau's definition by separating closure properties related to concrete constructions involving relations from local interpolation properties. We mention that for finite  $A$  the constructive part can be expressed via interpretations of primitive positive formulæ in both components. In fact, it was noted by Ágnes Szendrei that given a set  $Q \subseteq \text{Rp}_A$ , one may consider the relational structures  $\mathbf{A} = \langle A; (\varrho)_{(\varrho, \varrho') \in Q} \rangle$  and  $\mathbf{A}' = \langle A; (\varrho')_{(\varrho, \varrho') \in Q} \rangle$  and primitive positively definable relations on the product  $\mathbf{A} \times \mathbf{A}'$ : if  $\varphi$  is a primitive positive formula in the language of  $Q$  (including equality) with at most  $m$  free variables, then it defines the following  $m$ -ary relation on the product

$$\hat{\sigma} := \left\{ ((x_i, y_i))_{i \in m} \in (A^2)^m \mid (\mathbf{A} \times \mathbf{A}', ((x_i, y_i))_{i \in m}) \models \varphi \right\}$$

$$\begin{aligned}
&= \left\{ ((x_i, y_i))_{i \in m} \in (A^2)^m \mid (\underline{\mathbf{A}}, (x_i)_{i \in m}) \models \varphi \wedge (\underline{\mathbf{A}'}, (y_i)_{i \in m}) \models \varphi \right\} \\
&= \left\{ ((x_i, y_i))_{i \in m} \in (A^2)^m \mid ((x_i)_{i \in m}, (y_i)_{i \in m}) \in \sigma \times \sigma' \right\},
\end{aligned}$$

where  $\sigma := \{ \mathbf{x} \in A^m \mid (\underline{\mathbf{A}}, \mathbf{x}) \models \varphi \}$  and  $\sigma' := \{ \mathbf{x} \in A^m \mid (\underline{\mathbf{A}'}, \mathbf{x}) \models \varphi \}$ . If  $\sigma$  and  $\sigma'$  are both non-empty, then one may obtain the relation pair  $(\sigma, \sigma')$  defined by  $\varphi$  as projections of  $\hat{\sigma}$ . If one of them is the empty set, then  $\hat{\sigma} = \emptyset$  and therefore both projections will be empty. Thus only taking projections of  $\hat{\sigma}$  (i.e. of pp-definable relations in the product  $\underline{\mathbf{A}} \times \underline{\mathbf{A}'}$ ) will never produce relation pairs  $(\sigma, \sigma')$  where  $\sigma' = \emptyset \subsetneq \sigma$ , which is certainly needed, e.g., to model intersection in both components. However, collecting all pairs  $(\sigma, \sigma')$  arising from primitive positive formulæ  $\varphi$  correctly describes the closure  $[Q]_{\text{Rp}_A}$  in the case of finite carrier sets.

The second issue pointed out above is related to nullary operations. In the literature these are often neglected, which makes it necessary for relation pair algebras ([18]) and for relational clones (relation algebras, [29]) to contain the empty pair  $(\emptyset, \emptyset)$  and the empty relation, respectively, in order to be in accordance with the corresponding GALOIS theory.

If nullary operations are given their proper place, this absurdity vanishes (see [3] for clones and relational clones); then empty relations (pairs) get a true function, indicating by their presence the absence of nullary operations on the dual side (see Lemma 4.8 below). This is also the reason why we cannot and do not add the empty pairs of all arities as nullary constants to the closure condition of relation pair clones.

Relational clones (as given in [3, Definition 2.2, p. 8]) relate to relation pair clones in the following way:

**Lemma 4.4.** *For any carrier set  $A$  a subset  $Q \subseteq \text{Rp}_A$  is a relational clone if and only if  $P := \biguplus_{m \in \mathbb{N}} \{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \}$  is a relation pair clone.*

*Proof.* Evidently,  $P$  is closed under arbitrary general superpositions if and only if  $Q$  is.  $\square$

The following result is comparably easy.

**Lemma 4.5.** *Whenever  $Q \subseteq \text{Rp}_A$  is a relation pair clone on some set  $A$ , then*

$$\begin{aligned}
Q' &:= \left\{ \varrho \mid (\varrho, \varrho') \in Q^{(m)} \text{ for some } m \in \mathbb{N} \right\}, \\
Q'' &:= \left\{ \varrho' \mid (\varrho, \varrho') \in Q^{(m)} \text{ for some } m \in \mathbb{N} \right\} \text{ and} \\
Q''' &:= \left\{ \varrho \mid (\varrho, \varrho) \in Q^{(m)} \text{ for some } m \in \mathbb{N} \right\}
\end{aligned}$$

*are relational clones on  $A$ .*

*Proof.* Closedness of  $Q$  w.r.t. general superpositions carries over to  $Q'$ ,  $Q''$  and  $Q'''$ .  $\square$

Similarly as for semiclones, the GALOIS correspondence Polp-Invp provides many examples of relation pair clones (see [16, Lemma 9, p. 21] or [18, Lemma 9, p. 16] for the case of finite carrier sets; cf. [12, Lemma 3.1, p. 154] for the general framework of relational constraints and conjunctive minors).

**Lemma 4.6.** *For each  $F \subseteq \text{O}_A$  the set  $\text{Inv}_A F$  is a relation pair clone.*

*Proof.* To check that  $\text{Invp}_A F$  for  $F \subseteq O_A$  is closed w.r.t. to general superpositions let  $I$  and  $\mu$  be sets,  $m, m_i \in \mathbb{N}$  for  $i \in I$ , and  $\beta: m \rightarrow \mu$  and  $\alpha_i: m_i \rightarrow \mu$  for  $i \in I$  be mappings. For given relation pairs  $(\varrho_i, \varrho'_i) \in \text{Invp}_A^{(m_i)} F$  we are going to show that  $(\varrho, \varrho') := \bigwedge_{(\alpha_i)_{i \in I}} (\varrho_i, \varrho'_i)_{i \in I} \in \text{Invp}_A^{(m)} F$ . For this let  $f \in F$  and put  $n := \text{ar}(f)$ . To verify that  $f \triangleright (\varrho, \varrho')$ , let us take any  $\mathbf{r} \in \varrho^n$ . By definition of  $\varrho = \bigwedge_{(\alpha_i)_{i \in I}} (\varrho_i)_{i \in I}$ , for each  $0 \leq j < n$  there exists  $a_j \in A^\mu$  such that  $\mathbf{r}(j) = a_j \circ \beta$  and  $a_j \circ \alpha_i \in \varrho_i$  for all  $i \in I$ . By putting  $\mathbf{a} := (a_j)_{j \in n} \in (A^\mu)^n$ , we hence obtain  $\mathbf{r} = (a_j \circ \beta)_{j \in n} = \mathbf{a} \circ \beta$  (cp. Lemma 2.1(a)). Therefore, by associativity, we get  $f \circ (\mathbf{r}) = f \circ (\mathbf{a} \circ \beta) = (f \circ (\mathbf{a})) \circ \beta$ , which belongs to  $\varrho'$  as  $(f \circ (\mathbf{a})) \circ \alpha_i = f \circ (\mathbf{a} \circ \alpha_i) \in \varrho'_i$  for all  $i \in I$  (due to  $\mathbf{a} \circ \alpha_i = (a_j \circ \alpha_i)_{j \in n} \in \varrho_i^n$  and  $f$  preserving  $(\varrho_i, \varrho'_i) \in \text{Invp}_A^{(m_i)} F$ ).  $\square$

As a direct consequence we get the following compulsory corollary.

**Corollary 4.7.** *For any set  $Q \subseteq \text{Rp}_A$  we have*

$$[Q]_{\text{Rp}_A} \subseteq \text{Invp}_A \text{Polp}_A Q \quad \text{and} \quad \text{Polp}_A Q = \text{Polp}_A [Q]_{\text{Rp}_A}.$$

Next, we quickly address how nullary operations affect the associated relation pair algebras.

**Lemma 4.8.** *For  $F \subseteq O_A$  we have  $(\emptyset, \emptyset) \in \text{Invp}_A F$  if and only if  $F \subseteq O_A \setminus O_A^{(0)}$ .*

*Proof.* It is clear that every operation of positive arity preserves  $(\emptyset, \emptyset)$ , i.e. that  $(\emptyset, \emptyset) \in \text{Invp}_A (O_A \setminus O_A^{(0)})$ . Conversely, assume that  $F^{(0)} \neq \emptyset$ , say  $F$  contains a nullary constant operation  $c$  with value  $a \in A$ . If  $c \triangleright (\emptyset, \varrho)$  then it follows that  $(a, \dots, a) \in \varrho$ , i.e.  $\varrho \neq \emptyset$ . Thus,  $(\emptyset, \emptyset) \notin \text{Invp}_A \{c\}$ , and so  $(\emptyset, \emptyset) \notin \text{Invp}_A F$ .  $\square$

The following two results have their analogues in Proposition 1.11(a'),(b') from [29, p. 17].

**Lemma 4.9.** *For  $s \in \mathbb{N}$  and any set  $F \subseteq O_A^{(\leq s)} := \biguplus_{0 \leq n \leq s} O_A^{(n)}$  of at most  $s$ -ary operations, we have  $s\text{-LOC}_A \text{Invp}_A F = \text{Invp}_A F$ .*

*Proof.* Let  $m \in \mathbb{N}$  and  $(\sigma, \sigma') \in s\text{-LOC}_A^{(m)} \text{Invp}_A F$ . Consider any  $f \in F$ , then  $n := \text{ar}(f)$  necessarily fulfils  $n \leq s$ . Therefore, if we consider any  $\mathbf{r} = (r_j)_{0 \leq j < n}$  in  $\sigma^n$  and put  $B := \{r_j \mid 0 \leq j < n\} \subseteq \sigma \subseteq A^m$ , we clearly have a finite subset  $B \subseteq \sigma$  of cardinality at most  $n \leq s$ . As  $(\sigma, \sigma') \in s\text{-LOC}_A \text{Invp}_A F$ , there is some  $(\varrho, \varrho') \in \text{Invp}_A^{(m)} F$  such that  $B \subseteq \varrho$  and  $\varrho' \subseteq \sigma'$ . We know that  $f \triangleright (\varrho, \varrho')$ , so since  $B \subseteq \varrho$ , we get  $\mathbf{r} \in \varrho^n$  and thus  $f \circ (\mathbf{r}) \in \varrho' \subseteq \sigma'$ . Consequently, we have shown  $f \triangleright (\sigma, \sigma')$ , and as  $f \in F$  was arbitrary, we obtain  $(\sigma, \sigma') \in \text{Invp}_A F$  as desired.  $\square$

**Corollary 4.10.** *The equality  $\text{LOC}_A \text{Invp}_A F = \text{Invp}_A F$  is satisfied for any  $F \subseteq O_A$ .*

*Proof.* Consider  $(\sigma, \sigma') \in \text{LOC}_A \text{Invp}_A F$  and  $f \in F^{(n)}$ ,  $n \in \mathbb{N}$ . By definition of  $\text{LOC}_A$ , we have  $(\sigma, \sigma') \in n\text{-LOC}_A \text{Invp}_A F \subseteq n\text{-LOC}_A \text{Invp}_A \{f\}$ , which by Lemma 4.9 is equal to  $\text{Invp}_A \{f\}$ . As the function  $f \in F$  was arbitrarily chosen, we obtain  $(\sigma, \sigma') \in \text{Invp}_A F$ .  $\square$

## 5. CHARACTERISATION OF CLOSURES RELATED TO Polp-Invp

In this section we characterise, for any parameter  $s \in \mathbb{N}$ , the operators  $\text{Polp}_A \text{Invp}_A^{(\leq s)}$  and  $\text{Invp}_A \text{Polp}_A^{(\leq s)}$  as  $s$ -local closures of the generated semiclonal and relation pair clone, respectively. Subsequently, we present a few consequences of these theorems.

**5.1. The operational side.** For our task it is helpful to gather some knowledge about the least (w.r.t.  $\leq$  and thus a least among several equivalent ones w.r.t.  $\preceq$ ) pair  $(\varrho, \varrho') \in \text{Rp}_A^{(m)}$  being invariant for some set  $F \subseteq \text{O}_A$  and satisfying  $B \subseteq \varrho$  for a given finite set  $B \subseteq \text{R}_A^{(m)}$ ,  $m \in \mathbb{N}$ . Addressing this issue, the following lemma generalises Proposition 2.4 of [29, p. 21] from relations to relation pairs.

**Lemma 5.1.** *Let  $F \subseteq \text{O}_A$  be a set of operations and  $\mathbf{b} \in (A^m)^n$  for some  $m, n \in \mathbb{N}$ ; set  $B := \{\mathbf{b}(j) \mid 0 \leq j < n\} \subseteq A^m$ . Then the pair  $\Gamma_F(B) := (\varrho, \varrho')$ , where  $\varrho := \{f \circ (\mathbf{b}) \mid f \in \langle F \rangle_{\text{O}_A}^{(n)}\}$  and  $\varrho' := \{f \circ (\mathbf{b}) \mid f \in [F]_{\text{O}_A}^{(n)}\}$ , is the least pair (w.r.t.  $\leq$ ) in  $\text{Invp}_A^{(m)} F$  satisfying  $B \subseteq \varrho$ .*

Note that the lemma also shows that the relations  $\varrho, \varrho' \in \text{R}_A^{(m)}$  do not depend on the order of the entries of the tuple  $\mathbf{b}$ . Furthermore, instead of the finite cardinal  $m$ , any cardinal or, in fact, any indexing set  $K$  can be used, provided the notion of preservation is straightforwardly extended to relation pairs of arbitrary arity, i.e. pairs  $(R, S)$  such that  $S \subseteq R \subseteq A^K$ .

*Proof.* First of all, it is clear that  $B \subseteq \varrho$  as  $\text{J}_A^{(n)} \subseteq \langle F \rangle_{\text{O}_A}^{(n)}$ . Next, we prove that  $(\varrho, \varrho') \in \text{Invp}_A F$ . For this let  $\ell \in \mathbb{N}$ ,  $g \in F^{(\ell)}$  and  $\mathbf{r} = (r_j)_{0 \leq j < \ell} \in \varrho^\ell$ . By construction of  $\varrho$ , for each  $0 \leq j < \ell$  there exists some  $f_j \in \langle F \rangle_{\text{O}_A}^{(n)} = [F]_{\text{O}_A}^{(n)} \cup \text{J}_A^{(n)}$  (see Lemma 3.5(b)) such that  $r_j = f_j \circ (\mathbf{b})$ . Using Lemma 2.1(c), we have

$$g \circ (\mathbf{r}) = g \circ (f_0 \circ (\mathbf{b}), \dots, f_{\ell-1} \circ (\mathbf{b})) = (g \circ (f_0, \dots, f_{\ell-1})) \circ (\mathbf{b}) \in \varrho',$$

since  $g \circ (f_0, \dots, f_{\ell-1}) \in [F]_{\text{O}_A}^{(n)}$  by the closure property of semiclones.

Finally, we prove that any pair  $(\sigma, \sigma') \in \text{Invp}_A^{(m)} F$  satisfying  $B \subseteq \sigma$  fulfils  $(\varrho, \varrho') \leq (\sigma, \sigma')$ . By Corollary 3.8 we know  $(\sigma, \sigma') \in \text{Invp}_A F = \text{Invp}_A [F]_{\text{O}_A}$ , so since  $B \subseteq \sigma$  we have  $f \circ (\mathbf{b}) \in \sigma'$  for any  $f \in [F]_{\text{O}_A}^{(n)}$ . Therefore,  $\varrho' \subseteq \sigma' \subseteq \sigma$ . As, by Lemma 3.5(b),  $\langle F \rangle_{\text{O}_A}^{(n)} = [F]_{\text{O}_A}^{(n)} \cup \text{J}_A^{(n)}$ , it follows that  $\varrho = \varrho' \cup B$ . We have  $B \subseteq \sigma$  by assumption and  $\varrho' \subseteq \sigma$  as demonstrated before. Thus  $\varrho \subseteq \sigma$ , whence  $(\varrho, \varrho') \leq (\sigma, \sigma')$ .  $\square$

**Corollary 5.2.** *Let  $F \subseteq \text{O}_A$ ,  $n \in \mathbb{N}$ , and  $X \subseteq A^n$  be any subset of finite cardinality  $|X| =: k < \aleph_0$ ; moreover, consider an arbitrary bijection  $\beta: k \rightarrow X$  as fixed. Defining  $B := \{e_i^{(n)}|_X \circ \beta \mid 0 \leq i < n\} \subseteq A^k$ , as well as  $k$ -ary relations  $\varrho_{X,n} := \{f|_X \circ \beta \mid f \in \langle F \rangle_{\text{O}_A}^{(n)}\}$  and  $\varrho'_{X,n} := \{f|_X \circ \beta \mid f \in [F]_{\text{O}_A}^{(n)}\}$ , we have  $(\varrho_{X,n}, \varrho'_{X,n}) = \Gamma_F(B) \in \text{Invp}_A^{(k)} F$ .*

*Proof.* The claim follows from Lemma 5.1 by observing that the equality  $f \circ (e_i^{(n)}|_X)_{0 \leq i < n} = f \circ (e_i^{(n)})_{0 \leq i < n}|_X = f|_X$  holds for all  $f \in \text{O}_A^{(n)}$ .  $\square$

We are now prepared to prove our first theorem, characterising the closure  $\text{Polp}_A \text{Invp}_A^{(\leq s)}$  for  $s \in \mathbb{N}$ .

**Theorem 5.3.** *For  $s \in \mathbb{N}$  and any set of operations  $F \subseteq O_A$  we have the equality  $\text{Polp}_A \text{Invp}_A^{(\leq s)} F = s\text{-Loc}_A [F]_{O_A}$ .*

*Proof.* We have  $[F]_{O_A} \subseteq \text{Polp}_A \text{Invp}_A^{(\leq s)} [F]_{O_A} = \text{Polp}_A \text{Invp}_A^{(\leq s)} F$  by Corollary 3.8, whence  $s\text{-Loc}_A [F]_{O_A} \subseteq s\text{-Loc}_A \text{Polp}_A \text{Invp}_A^{(\leq s)} F = \text{Polp}_A \text{Invp}_A^{(\leq s)} F$ , using Lemma 3.10.

For the converse inclusion take  $g \in \text{Polp}_A^{(n)} \text{Invp}_A^{(\leq s)} F$  for any  $n \in \mathbb{N}$ ; we want to prove that  $g \in s\text{-Loc}_A^{(n)} [F]_{O_A}$ . To do so, we consider any finite  $X \subseteq A^n$  where  $k := |X| \leq s$  and an arbitrary bijection  $\beta: k \rightarrow X$ . Now Corollary 5.2 yields that  $(\varrho_{X,n}, \varrho'_{X,n}) \in \text{Invp}_A^{(k)} F \subseteq \text{Invp}_A^{(\leq s)} F$ , wherefore  $g \triangleright (\varrho_{X,n}, \varrho'_{X,n})$ . Moreover, we have  $B = \left\{ e_i^{(n)}|_X \circ \beta \mid 0 \leq i < n \right\} \subseteq \varrho_{X,n}$ , whence we obtain  $g|_X \circ \beta = g \circ \left( e_i^{(n)}|_X \circ \beta \right)_{0 \leq i < n} = g \circ \left( e_i^{(n)}|_X \circ \beta \right)_{0 \leq i < n} \in \varrho'_{X,n}$ . Thus by definition of  $\varrho'_{X,n}$  there has to exist some  $f \in [F]_{O_A}^{(n)}$  such that  $g|_X \circ \beta = f|_X \circ \beta$ , which implies  $g|_X = f|_X$  by bijectivity of  $\beta$ . Yet, this finally proves that  $g \in s\text{-Loc}_A^{(n)} [F]_{O_A}$ .  $\square$

The following simple observation is not unexpected.

**Lemma 5.4.** *Any relation pair clone  $Q \subseteq \text{Rp}_A$  on a non-empty carrier set  $A$  satisfies  $Q^{(m)} \subseteq [Q^{(s)}]_{\text{Rp}_A}$  for all  $m, s \in \mathbb{N}$  where  $m \leq s$ .*

*For  $A = \emptyset$ , we have in fact  $Q^{(s)} \subseteq [Q^{(0)}]_{\text{Rp}_A}$  for all  $s \in \mathbb{N}$  and any relation pair clone  $Q \subseteq \text{Rp}_A$ .*

*Proof.* For  $A \neq \emptyset$ , it is clear for  $m$ -ary relations  $\varrho \subseteq A^m$  that one can write  $\varrho = \text{pr}_{0,\dots,m-1} (\varrho \times A^{s-m})$ . Designating by  $\iota: m \rightarrow s$  the identical embedding, one may rewrite this relationship as  $\varrho = \bigwedge_{\text{id}_s}^\iota \bigwedge_{\text{id}_s}^{\text{id}_s} \varrho$ . Since the definition of the operators only depends on the arity of the relation  $\varrho$ , the same works for  $m$ -ary relation pairs. So, if  $(\varrho, \varrho') \in Q^{(m)}$ , then  $\bigwedge_{\text{id}_s}^{\text{id}_s} (\varrho, \varrho') \in Q^{(s)}$ , and thus  $(\varrho, \varrho') = \bigwedge_{\text{id}_s}^\iota \bigwedge_{\text{id}_s}^{\text{id}_s} (\varrho, \varrho') \in [Q^{(s)}]_{\text{Rp}_A}$ .

For the empty carrier set, in point of fact, the opposite holds: for  $s = 0$  the claim is trivial. For  $s \in \mathbb{N}_+$  and any  $\varrho \in \text{R}_A^{(s)}$ , we have  $\varrho = A^s = \emptyset$ , and  $\varrho = \emptyset \times \emptyset = (\text{pr}_\emptyset \varrho) \times A^s = \bigwedge_{\text{id}_s}^{\text{id}_s} \bigwedge_{\text{id}_s}^\iota \varrho$  where  $\iota$  is the map from above for  $m = 0$ . Since  $\bigwedge_{\text{id}_s}^\iota (\varrho, \varrho') \in Q^{(0)}$  for  $(\varrho, \varrho') \in Q^{(s)}$ , we have  $Q^{(s)} \subseteq [Q^{(0)}]_{\text{Rp}_A}$ .  $\square$

Hence, we can prove the first corollary to our theorem.

**Corollary 5.5.** *For  $s \in \mathbb{N}$  and any set of operations  $F \subseteq O_A$  on  $A \neq \emptyset$  we have the equality  $\text{Polp}_A \text{Invp}_A^{(s)} F = s\text{-Loc}_A [F]_{O_A}$ .*

*If  $A = \emptyset$ , we have  $s\text{-Loc}_A [F]_{O_A} = \text{Polp}_A \text{Invp}_A^{(0)} F = 0\text{-Loc}_A [F]_{O_A}$  for any  $F \subseteq O_A$  and  $s \in \mathbb{N}$ .*

*Proof.* By Lemma 4.6, the set  $\text{Invp}_A F$  is a relation pair clone, so Lemma 5.4 yields  $\text{Invp}_A^{(m)} F \subseteq [\text{Invp}_A^{(s)} F]_{\text{Rp}_A}$  for all  $m \leq s$  and  $A \neq \emptyset$ . From Corollary 4.7

one obtains  $\text{Polp}_A \text{Invp}_A^{(m)} F \supseteq \text{Polp}_A \left[ \text{Invp}_A^{(s)} F \right]_{\text{Rp}_A} = \text{Polp}_A \text{Invp}_A^{(s)} F$  for all  $m \leq s$ , and hence, we have

$$\begin{aligned} \text{Polp}_A \text{Invp}_A^{(s)} F &= \bigcap_{0 \leq m \leq s} \text{Polp}_A \text{Invp}_A^{(m)} F = \text{Polp}_A \bigcup_{0 \leq m \leq s} \text{Invp}_A^{(m)} F \\ &= \text{Polp}_A \text{Invp}_A^{(\leq s)} F = s\text{-Loc}_A [F]_{O_A}, \end{aligned}$$

where the last equality holds by Theorem 5.3.

The claim for  $A = \emptyset$  follows by similar transformations.  $\square$

The second corollary characterises the closure  $\text{Polp}_A \text{Invp}_A$ .

**Corollary 5.6.** *We have  $\text{Polp}_A \text{Invp}_A F = \text{Loc}_A [F]_{O_A}$  for all  $F \subseteq O_A$ .*

*Proof.* Using the definition of the operator  $\text{Loc}_A$  and Theorem 5.3, we can write

$$\begin{aligned} \text{Loc}_A [F]_{O_A} &= \bigcap_{s \in \mathbb{N}} s\text{-Loc}_A [F]_{O_A} = \bigcap_{s \in \mathbb{N}} \text{Polp}_A \text{Invp}_A^{(\leq s)} F \\ &= \text{Polp}_A \bigcup_{s \in \mathbb{N}} \text{Invp}_A^{(\leq s)} F = \text{Polp}_A \text{Invp}_A F. \quad \square \end{aligned}$$

The third corollary proves that a set  $F \subseteq O_A$  is closed w.r.t.  $[\ ]_{O_A}$  and  $s\text{-Loc}_A$  if (and clearly only if) it is closed w.r.t. to the operator  $s\text{-Loc}_A [\ ]_{O_A}$ . An analogous result holds, of course, for the operators  $[\ ]_{O_A}$ ,  $\text{Loc}_A$  and  $\text{Loc}_A [\ ]_{O_A}$ . These two facts can be seen as generalisations of Lemma 2.5(ii),(iii) in [29, p. 22], where similar results have been proven for clones.

**Corollary 5.7.** *For  $s \in \mathbb{N}$  a set  $F \subseteq O_A$  of operations is an  $s$ -locally (locally) closed semiclone if and only if  $s\text{-Loc}_A [F]_{O_A} = F$  ( $\text{Loc}_A [F]_{O_A} = F$ ).*

*Proof.* If  $[F]_{O_A} = F$  and  $s\text{-Loc}_A F = F$  ( $\text{Loc}_A F = F$ ), then it clearly follows that  $s\text{-Loc}_A [F]_{O_A} = F$  ( $\text{Loc}_A [F]_{O_A} = F$ ). Conversely, if the latter equality holds, then we obviously have  $s\text{-Loc}_A F = F$  ( $\text{Loc}_A F = F$ ) by idempotence of the  $s$ -local (local) closure. Besides, combining the condition  $F = s\text{-Loc}_A [F]_{O_A}$  ( $F = \text{Loc}_A [F]_{O_A}$ ) with Theorem 5.3 (Corollary 5.6), one obtains the equality  $F = \text{Polp}_A \text{Invp}_A^{(\leq s)} F$  ( $F = \text{Polp}_A \text{Invp}_A F$ ), and the latter set is a semiclone by Lemma 3.7. Therefore,  $[F]_{O_A} = F$ .  $\square$

The next corollary is in analogy to Lemma 2.6 in [29, p. 22].

**Corollary 5.8.** *For every  $F \subseteq O_A$  we have*

- (a)  $\text{Invp}_A^{(m)} F = \text{Invp}_A^{(m)} [F]_{O_A} = \text{Invp}_A^{(m)} \text{Loc}_A [F]_{O_A} = \text{Invp}_A^{(m)} s\text{-Loc}_A [F]_{O_A}$   
for all  $m \leq s \in \mathbb{N}$  whenever  $A \neq \emptyset$ .
- (b)  $\text{Invp}_A F = \text{Invp}_A [F]_{O_A} = \text{Invp}_A \text{Loc}_A [F]_{O_A}$ .

*Proof.* (a) By Corollaries 5.5 and 5.7, and since  $m \leq s$ , we have

$$\begin{aligned} \text{Polp}_A \text{Invp}_A^{(m)} s\text{-Loc}_A [F]_{O_A} &= m\text{-Loc}_A [s\text{-Loc}_A [F]_{O_A}]_{O_A} \\ &= m\text{-Loc}_A s\text{-Loc}_A [F]_{O_A} = m\text{-Loc}_A [F]_{O_A} = \text{Polp}_A \text{Invp}_A^{(m)} F, \end{aligned}$$

which implies  $\text{Invp}_A^{(m)} s\text{-Loc}_A [F]_{O_A} = \text{Invp}_A^{(m)} F$  by applying  $\text{Invp}_A^{(m)}$  once more on both sides.

- (b) Similarly, we have  $\text{Invp}_A \text{Loc}_A [F]_{\text{O}_A} = \text{Invp}_A \text{Polp}_A \text{Invp}_A F = \text{Invp}_A F$ , using Corollary 5.6.  $\square$

Next, we turn to the characterisation of the other part of the GALOIS connection.

**5.2. The side of relation pairs.** We start by preparing the proof of our theorem with a lemma.

**Lemma 5.9.** *Let  $Q \subseteq \text{Rp}_A$  be any set of relation pairs,  $m \in \mathbb{N}$  an arity and  $B \subseteq A^m$  be a finite subset of cardinality  $n := |B|$ . Consider any enumeration  $\mathbf{b} = (b_0, \dots, b_{s-1}) \in B^s$  of  $B = \{b_0, \dots, b_{s-1}\}$  ( $s \geq n$ ) and define*

$$\mu'_B := \left\{ f \circ (\mathbf{b}) \mid f \in \text{Polp}_A^{(s)} Q \right\}, \quad \mu_B := \left\{ f \circ (\mathbf{b}) \mid f \in \text{Pol}_A^{(s)} Q_1 \right\},$$

where  $Q_1 := \{ \varrho \in \text{R}_A \mid (\varrho, \varrho') \in Q \}$ .

- (a) *The pair  $(\mu_B, \mu'_B)$  can be obtained from  $Q$  by general superpositions, i.e.  $(\mu_B, \mu'_B) \in [Q]_{\text{Rp}_A}$ .*  
 (b) *For  $F := \text{Polp}_A Q$  one may obtain  $\Gamma_F(B)$  as a relaxation of  $(\mu_B, \mu'_B)$ , that is,  $\Gamma_F(B) \in {}^\rightarrow [Q]_{\text{Rp}_A}^\leftarrow$ .*

*Proof.* (a) In order to prove that  $(\mu_B, \mu'_B) \in [Q]_{\text{Rp}_A}$ , we shall exhibit a general composition producing this relation pair from the ones in  $Q$ . Using the notation from Definition 4.2, we choose  $\mu := A^s$  and define  $\beta: m \rightarrow A^s$  by  $\beta(i) := (b_0(i), \dots, b_{s-1}(i))$  for  $0 \leq i < m$ . Moreover, for  $n \in \mathbb{N}$  and  $(\varrho, \varrho') \in Q^{(n)}$  we put  $I_{n,(\varrho, \varrho')} := \{ (n, \varrho, \varrho', \mathbf{r}) \mid \mathbf{r} \in \varrho^s \}$ ; further, we define  $I := \biguplus_{n \in \mathbb{N}} \bigcup_{(\varrho, \varrho') \in Q^{(n)}} I_{n,(\varrho, \varrho')}$ . Finally, for  $(n, \varrho, \varrho', \mathbf{r}) \in I$  let the function  $\alpha_{n, \varrho, \varrho', \mathbf{r}}: n \rightarrow A^s$  be given by  $\alpha_{n, \varrho, \varrho', \mathbf{r}}(j) := (r_0(j), \dots, r_{s-1}(j))$  for  $0 \leq j < n$ , where  $\mathbf{r} = (r_0, \dots, r_{s-1}) \in \varrho^s$ .

We claim now that  $(\mu_B, \mu'_B) = \bigwedge_{(\alpha_{n, \varrho, \varrho', \mathbf{r}})_{(n, \varrho, \varrho', \mathbf{r}) \in I}}^\beta (\varrho, \varrho')_{(n, \varrho, \varrho', \mathbf{r}) \in I}$ ,

which can be checked by the following straightforward calculation. Denoting for each  $(n, \varrho, \varrho', \mathbf{r}) \in I$  by  $\sigma_{n, \varrho, \varrho', \mathbf{r}}$  some relation in  $\text{R}_A^{(n)}$ , we have

$$\begin{aligned} & \bigwedge_{(\alpha_{n, \varrho, \varrho', \mathbf{r}})_{(n, \varrho, \varrho', \mathbf{r}) \in I}}^\beta (\sigma_{n, \varrho, \varrho', \mathbf{r}})_{(n, \varrho, \varrho', \mathbf{r}) \in I} = \\ &= \left\{ (f(\beta(i)))_{0 \leq i < m} \mid \begin{array}{l} f \in A^{A^s} \wedge \forall (n, \varrho, \varrho', \mathbf{r}) \in I: \\ (f(\alpha_{n, \varrho, \varrho', \mathbf{r}}(0)), \dots, f(\alpha_{n, \varrho, \varrho', \mathbf{r}}(n-1))) \in \sigma_{n, \varrho, \varrho', \mathbf{r}} \end{array} \right\} \\ &= \left\{ f \circ (b_0, \dots, b_{s-1}) \mid \begin{array}{l} f \in A^{A^s} \wedge \forall n \in \mathbb{N} \forall (\varrho, \varrho') \in Q^{(n)} \\ \forall \mathbf{r} = (r_0, \dots, r_{s-1}) \in \varrho^s: \\ f \circ (r_0, \dots, r_{s-1}) \in \sigma_{n, \varrho, \varrho', \mathbf{r}} \end{array} \right\} \\ &= \left\{ f \circ (b_0, \dots, b_{s-1}) \mid f \in A^{A^s} \wedge \forall n \in \mathbb{N} \forall (\varrho, \varrho') \in Q^{(n)}: f \triangleright (\varrho, \sigma_{n, \varrho, \varrho', \mathbf{r}}) \right\}. \end{aligned}$$

Specialising this to  $\sigma_{n, \varrho, \varrho', \mathbf{r}} := \varrho'$ , we get

$$\bigwedge_{(\alpha_{n, \varrho, \varrho', \mathbf{r}})_{(n, \varrho, \varrho', \mathbf{r}) \in I}}^\beta (\varrho')_{(n, \varrho, \varrho', \mathbf{r}) \in I} =$$



$$\begin{aligned}
&= \left\{ f \circ (b_0, \dots, b_{s-1}) \mid f \in A^{A^s} \wedge \forall n \in \mathbb{N} \forall (\varrho, \varrho') \in Q^{(n)} : f \triangleright (\varrho, \varrho') \right\} \\
&= \left\{ f \circ (b_0, \dots, b_{s-1}) \mid f \in \text{Pol}_A^{(s)} Q \right\} = \mu'_B.
\end{aligned}$$

Specialising once more to  $\sigma_{n, \varrho, \varrho', \mathbf{r}} := \varrho$ , we obtain

$$\begin{aligned}
&\bigwedge_{\beta} (\varrho)_{(n, \varrho, \varrho', \mathbf{r}) \in I} = \\
&(\alpha_{n, \varrho, \varrho', \mathbf{r}})_{(n, \varrho, \varrho', \mathbf{r}) \in I} = \\
&= \left\{ f \circ (b_0, \dots, b_{s-1}) \mid f \in A^{A^s} \wedge \forall n \in \mathbb{N} \forall (\varrho, \varrho') \in Q^{(n)} : f \triangleright (\varrho, \varrho') \right\} \\
&= \left\{ f \circ (b_0, \dots, b_{s-1}) \mid f \in \text{Pol}_A^{(s)} Q_1 \right\} = \mu_B.
\end{aligned}$$

- (b) By Lemma 3.7 we have  $[F]_{O_A} = F$ , and therefore Lemma 3.5(b) yields  $\langle F \rangle_{O_A} = [F]_{O_A} \cup J_A = F \cup J_A$ . Hence, according to Lemma 5.1, we obtain that  $\Gamma_F(B) = (\varrho, \varrho')$ , wherein  $\varrho = \left\{ f \circ (\mathbf{b}) \mid f \in F^{(s)} \cup J_A^{(s)} \right\}$  and  $\varrho' = \left\{ f \circ (\mathbf{b}) \mid f \in F^{(s)} \right\} = \left\{ f \circ (\mathbf{b}) \mid f \in \text{Pol}_A^{(s)} Q \right\} = \mu'_B$ . Moreover, as obviously  $F = \text{Pol}_A Q \subseteq \text{Pol}_A Q_1$ , we have  $\mu'_B = \varrho' \subseteq \varrho \subseteq \mu_B$ . Since  $(\mu_B, \mu'_B) \in [Q]_{\text{Rp}_A}$  by (a), we finally see  $\Gamma_F(B) = (\varrho, \varrho') \in \rightarrow[Q]_{\text{Rp}_A}^{\leftarrow}$ .  $\square$

Now, we are ready to address the dual side of the GALOIS correspondence, i.e. the closure  $\text{Inv}_A \text{Pol}_A^{(\leq s)}$ .

**Theorem 5.10.** *For  $s \in \mathbb{N}$  and any set  $Q \subseteq \text{Rp}_A$  of relation pairs we have  $\text{Inv}_A \text{Pol}_A^{(\leq s)} Q = s\text{-LOC}_A [Q]_{\text{Rp}_A}$ .*

*Proof.* We have  $[Q]_{\text{Rp}_A} \subseteq \text{Inv}_A \text{Pol}_A^{(\leq s)} [Q]_{\text{Rp}_A} = \text{Inv}_A \text{Pol}_A^{(\leq s)} Q$  by Corollary 4.7, so  $s\text{-LOC}_A [Q]_{\text{Rp}_A} \subseteq s\text{-LOC}_A \text{Inv}_A \text{Pol}_A^{(\leq s)} Q = \text{Inv}_A \text{Pol}_A^{(\leq s)} Q$  by Lemma 4.9.

For the converse inclusion let us consider  $m \in \mathbb{N}$  and an arbitrary  $m$ -ary pair  $(\sigma, \sigma') \in \text{Inv}_A^{(m)} \text{Pol}_A^{(\leq s)} Q$ . In order to prove that  $(\sigma, \sigma') \in s\text{-LOC}_A [Q]_{\text{Rp}_A}$ , we take any subset  $B \subseteq \sigma$  such that  $|B| \leq s$ . From Lemma 5.9(a) we get  $(\mu_B, \mu'_B) \in [Q]_{\text{Rp}_A}$ , and obviously we have  $B \subseteq \mu_B$ . Moreover, since  $(\sigma, \sigma')$  belongs to  $\text{Inv}_A \text{Pol}_A^{(\leq s)} Q$ , we have  $f \triangleright (\sigma, \sigma')$  for all  $f \in \text{Pol}_A^{(\leq s)} Q$ . So, as  $|B| \leq s$  and  $B \subseteq \sigma$ , we get  $\mu'_B \subseteq \sigma'$ . This proves  $(\sigma, \sigma') \in s\text{-LOC}_A [Q]_{\text{Rp}_A}$ .  $\square$

The following result is the analogue of Lemma 5.4 for semiclones.

**Lemma 5.11.** *Any semiclone  $F \subseteq O_A$  satisfies  $F^{(n)} \subseteq [F^{(s)}]_{O_A}$  for all arities  $n, s \in \mathbb{N}$  where  $0 < n \leq s$ .*

*Proof.* For  $f \in F^{(n)}$  we have  $g := f \circ (e_0^{(s)}, \dots, e_{n-1}^{(s)}) \in [F]_{O_A}^{(s)} = F^{(s)}$ . It follows that  $f = g \circ (e_0^{(n)}, \dots, e_{n-1}^{(n)}, e_{n-1}^{(n)}, \dots, e_{n-1}^{(n)}) \in [g]_{O_A} \subseteq [F^{(s)}]_{O_A}$  due to superassociativity. Hence, we obtain  $F^{(n)} \subseteq [F^{(s)}]_{O_A}$ .  $\square$

For  $n = 0 < s$  the previous lemma (and its proof) fail. This is why in the following corollary to Theorem 5.10 arities  $s$  and  $0$  are required.

**Corollary 5.12.** *For  $s \in \mathbb{N}$  and any set  $Q \subseteq R_A$  of relation pairs we have the equality  $\text{Inv}_A \text{Pol}_A^{(0, s)} Q = s\text{-LOC}_A [Q]_{\text{Rp}_A}$ .*

*Proof.* As, by Lemma 3.7, the set  $\text{Polp}_A Q$  is a semicclone, Lemma 5.11 is applicable and yields  $\text{Polp}_A^{(n)} Q \subseteq [\text{Polp}_A^{(s)} Q]_{O_A}$  for all  $0 < n \leq s$ . Via Corollary 3.8 this implies  $\text{Invp}_A \text{Polp}_A^{(n)} Q \supseteq \text{Invp}_A [\text{Polp}_A^{(s)} Q]_{O_A} = \text{Invp}_A \text{Polp}_A^{(s)} Q$  for all  $0 < n \leq s$ , whence we obtain

$$\begin{aligned} \text{Invp}_A \left( \text{Polp}_A^{(0)} Q \uplus \text{Polp}_A^{(s)} Q \right) &= \text{Invp}_A \text{Polp}_A^{(0)} Q \cap \text{Invp}_A \text{Polp}_A^{(s)} Q \\ &= \bigcap_{0 \leq n \leq s} \text{Invp}_A \text{Polp}_A^{(n)} Q \\ &= \text{Invp}_A \bigoplus_{0 \leq n \leq s} \text{Polp}_A^{(n)} Q = \text{Invp}_A \text{Polp}_A^{(\leq s)} Q \\ &= s\text{-LOC}_A [Q]_{\text{Rp}_A}, \end{aligned}$$

where the last equality is true by Theorem 5.10.  $\square$

In case that the relation pairs contain an empty pair, the nullary polymorphisms in Corollary 5.12 vanish.

**Corollary 5.13.** *For  $s, m \in \mathbb{N}$  and any set  $Q \subseteq \text{Rp}_A$  such that  $(\emptyset, \emptyset) \in Q^{(m)}$ , we have  $\text{Invp}_A \text{Polp}_A^{(s)} Q = s\text{-LOC}_A [Q]_{\text{Rp}_A}$ .*

*Proof.* Since a pair of empty relations belongs to  $Q$ , and thus to  $\text{Invp}_A \text{Polp}_A Q$ , Lemma 4.8 instantiated for  $F = \text{Polp}_A Q$  implies that  $\text{Polp}_A Q \subseteq O_A \setminus O_A^{(0)}$ , i.e.  $\text{Polp}_A^{(0)} Q = \emptyset$ . Therefore, the claim follows from Corollary 5.12.  $\square$

Next, we characterise the closure  $\text{Invp}_A \text{Polp}_A$ .

**Corollary 5.14.** *We have  $\text{Invp}_A \text{Polp}_A Q = \text{LOC}_A [Q]_{\text{Rp}_A}$  for all  $Q \subseteq \text{Rp}_A$ .*

*Proof.* From the definition of the operator  $\text{LOC}_A$  and Theorem 5.10, we obtain

$$\begin{aligned} \text{LOC}_A [Q]_{\text{Rp}_A} &= \bigcap_{s \in \mathbb{N}} s\text{-LOC}_A [Q]_{\text{Rp}_A} = \bigcap_{s \in \mathbb{N}} \text{Invp}_A \text{Polp}_A^{(\leq s)} Q \\ &= \text{Invp}_A \bigoplus_{s \in \mathbb{N}} \text{Polp}_A^{(\leq s)} Q = \text{Invp}_A \text{Polp}_A Q. \end{aligned} \quad \square$$

Moreover, we can infer that a set  $Q \subseteq \text{Rp}_A$  is closed w.r.t.  $[\ ]_{\text{Rp}_A}$  and  $s\text{-LOC}_A$  if (and clearly only if) it is closed w.r.t. to the operator  $s\text{-LOC}_A [\ ]_{\text{Rp}_A}$ . An analogous result holds, of course, for the operators  $[\ ]_{\text{Rp}_A}$ ,  $\text{LOC}_A$  and  $\text{LOC}_A [\ ]_{\text{Rp}_A}$ . These two facts can be seen to generalise Proposition 3.8(ii),(iii) in [29, p. 30], where similar statements have been proven for relational clones.

**Corollary 5.15.** *For  $s \in \mathbb{N}$  a set  $Q \subseteq \text{Rp}_A$  of relation pairs is an  $s$ -locally (locally) closed relation pair clone if and only if  $s\text{-LOC}_A [Q]_{\text{Rp}_A} = Q$  ( $\text{LOC}_A [Q]_{\text{Rp}_A} = Q$ ).*

*Proof.* Suppose that  $[Q]_{\text{Rp}_A} = Q$  and  $s\text{-LOC}_A Q = Q$  ( $\text{LOC}_A Q = Q$ ), then it follows evidently that  $s\text{-LOC}_A [Q]_{\text{Rp}_A} = Q$  ( $\text{LOC}_A [Q]_{\text{Rp}_A} = Q$ ). Conversely, assume the latter equality to hold. By idempotence of  $s\text{-LOC}_A$  ( $\text{LOC}_A$ ), we clearly get  $s\text{-LOC}_A Q = Q$  ( $\text{LOC}_A Q = Q$ ). Moreover, if we combine our assumption  $Q = s\text{-LOC}_A [Q]_{\text{Rp}_A}$  ( $Q = \text{LOC}_A [Q]_{\text{Rp}_A}$ ) with Theorem 5.10 (Corollary 5.14) we obtain  $Q = \text{Invp}_A \text{Polp}_A^{(\leq s)} Q$  ( $Q = \text{Invp}_A \text{Polp}_A Q$ ), and the latter set always is a relation pair clone by Lemma 4.6. Hence,  $[Q]_{\text{Rp}_A} = Q$ .  $\square$

The next corollary is the analogue of Proposition 3.9 in [29, p. 30 et seq.].

**Corollary 5.16.** *For every  $Q \subseteq \text{Rp}_A$  we have*

- (a) *that the set  $\text{Polp}_A^{(0,n)} Q = \text{Polp}_A^{(0,n)} [Q]_{\text{Rp}_A} = \text{Polp}_A^{(0,n)} \text{LOC}_A [Q]_{\text{Rp}_A}$  equals  $\text{Polp}_A^{(0,n)} s\text{-LOC}_A [Q]_{\text{Rp}_A}$  for all  $0 \leq n \leq s \in \mathbb{N}$ .*
- (b) *that  $\text{Polp}_A^{(n)} Q = \text{Polp}_A^{(n)} s\text{-LOC}_A [Q]_{\text{Rp}_A}$  for all  $0 \leq n \leq s \in \mathbb{N}$ , whenever  $(\emptyset, \emptyset) \in Q^{(m)}$  for some arity  $m \in \mathbb{N}$ .*
- (c)  *$\text{Polp}_A Q = \text{Polp}_A [Q]_{\text{Rp}_A} = \text{Polp}_A \text{LOC}_A [Q]_{\text{Rp}_A}$ .*

*Proof.* (a) It suffices to prove that  $\text{Polp}_A^{(0,n)} Q \subseteq \text{Polp}_A^{(0,n)} s\text{-LOC}_A [Q]_{\text{Rp}_A}$  is true for all  $n \leq s$ . Upon application of Corollary 5.12, we can infer that  $\text{Invp}_A \text{Polp}_A^{(0,n)} s\text{-LOC}_A [Q]_{\text{Rp}_A} = n\text{-LOC}_A [s\text{-LOC}_A [Q]_{\text{Rp}_A}]_{\text{Rp}_A}$ , which by Corollary 5.15 equals  $n\text{-LOC}_A s\text{-LOC}_A [Q]_{\text{Rp}_A}$ . Due to  $n \leq s$  the latter set coincides with  $n\text{-LOC}_A [Q]_{\text{Rp}_A}$ , which is  $\text{Invp}_A \text{Polp}_A^{(0,n)} Q$ , using again Corollary 5.12. Now the claim follows by once more applying  $\text{Polp}_A^{(0,n)}$  to both sides of the equation.

- (b) If  $(\emptyset, \emptyset) \in Q^{(m)}$  for some  $m \in \mathbb{N}$ , then we may substitute in the proof of (a) the operator  $\text{Polp}_A^{(0,n)}$  by  $\text{Polp}_A^{(n)}$  and the use of Corollary 5.12 by Corollary 5.13. Everything else works as just seen.
- (c) We have  $\text{Polp}_A \text{LOC}_A [Q]_{\text{Rp}_A} = \text{Polp}_A \text{Invp}_A \text{Polp}_A Q = \text{Polp}_A Q$  by Corollary 5.14; the other inclusions are trivial.  $\square$

**5.3. Characterisation of local closures for relation pairs.** Finally, we shall consider another characterisation of the  $s$ -local closure operators, involving  $s$ -directed unions. The statement can be improved for sets of relation pairs fulfilling an additional closure property, which is in particular satisfied by relation pair clones. Hence, our characterisation is especially useful in connection with the operator  $\text{LOC}_A [\cdot]_{\text{Rp}_A}$ .

Our first result is a generalisation of Proposition 1.13(ii) in [29, p. 18] to relation pairs (see also [28, Proposition 1.6(ii), p. 256]). It works provided one accepts the axiom of choice.

**Proposition 5.17.** *For any set  $Q \subseteq \text{Rp}_A$  of relation pairs and all  $s \in \mathbb{N}_+$ , the following holds:*

$$s\text{-LOC}_A Q = \biguplus_{m \in \mathbb{N}} \left\{ (\sigma, \sigma') \in \text{Rp}_A^{(m)} \left| \begin{array}{l} \exists \sigma'' \in \text{R}_A^{(m)} : \sigma'' \subseteq \sigma' \wedge \\ \exists \mathcal{T} \subseteq \text{Rp}_A^{(m)} \text{ } s\text{-directed} : \\ (\sigma, \sigma'') = \bigcup \mathcal{T} \wedge \\ \forall (\varrho, \varrho') \in \mathcal{T} \exists \tilde{\varrho} \in \text{R}_A^{(m)} : \\ \varrho \subseteq \tilde{\varrho} \wedge (\tilde{\varrho}, \varrho') \in Q^{(m)} \end{array} \right. \right\}.$$

*Proof.* To prove the inclusion “ $\supseteq$ ”, let us consider any  $m \in \mathbb{N}$  and a pair  $(\sigma, \sigma') \in \text{Rp}_A^{(m)}$  satisfying the lengthy condition in the proposition. Its first part says that there is an  $s$ -directed system  $\mathcal{T} \subseteq \text{Rp}_A^{(m)}$  whose union equals  $(\sigma, \sigma'')$  for some  $m$ -ary relation  $\sigma'' \subseteq \sigma'$ . The remaining part states that for every pair  $(\varrho, \varrho') \in \mathcal{T}$  there is an  $m$ -ary relation  $\tilde{\varrho} \supseteq \varrho$  such that  $(\tilde{\varrho}, \varrho') \in Q^{(m)}$ .

This implies that  $(\varrho, \varrho') \in \rightarrow \{(\tilde{\varrho}, \varrho')\}^\leftarrow \subseteq \rightarrow Q^\leftarrow \subseteq \rightarrow s\text{-LOC}_A Q^\leftarrow$ . Since the set  $s\text{-LOC}_A Q$  is  $s$ -locally closed, Corollary 2.11 implies that it is also closed under relaxation. Hence, we have  $\mathcal{T} \subseteq \rightarrow s\text{-LOC}_A Q^\leftarrow = s\text{-LOC}_A Q$ . Now as  $\mathcal{T}$  is an  $s$ -directed system, Lemma 2.14 yields that  $(\sigma, \sigma'') \in s\text{-LOC}_A Q$ . Thus, from  $\sigma'' \subseteq \sigma' \subseteq \sigma = \sigma$  we get that  $(\sigma, \sigma') \in \rightarrow \{(\sigma, \sigma'')\}^\leftarrow \subseteq \rightarrow s\text{-LOC}_A Q^\leftarrow$ , which we already showed to coincide with  $s\text{-LOC}_A Q$ .

For the converse inclusion, take any  $(\sigma, \sigma') \in s\text{-LOC}_A^{(m)} Q$ ,  $m \in \mathbb{N}$ . Then for any  $B \subseteq \sigma$  such that  $|B| \leq s$ , the set  $\Sigma_B := \{(\varrho, \varrho') \in Q^{(m)} \mid B \subseteq \varrho \wedge \varrho' \subseteq \sigma'\}$  is non-empty; using the axiom of choice, one can fix some pair  $(\tilde{\varrho}_B, \varrho'_B) \in \Sigma_B$ . It satisfies  $\varrho'_B \subseteq \sigma' \subseteq \sigma$ , thus,  $\varrho'_B \subseteq \tilde{\varrho}_B \cap \sigma =: \varrho_B$ . By construction, we have  $B \subseteq \varrho_B \subseteq \tilde{\varrho}_B$ , thus putting  $\mathcal{T} := \{(\varrho_B, \varrho'_B) \mid B \subseteq \sigma \wedge |B| \leq s\}$ , the collection  $\mathcal{T}$  satisfies the second part of the condition we need to verify. We shall check that  $\mathcal{T}$  is  $s$ -directed farther below; first we deal with the union  $(\mu, \mu') := \bigcup \mathcal{T}$  (meaning union in both components). Since for every subset  $B \subseteq \sigma$ ,  $|B| \leq s$ , we have  $\varrho'_B \subseteq \sigma'$  and  $\varrho_B \subseteq \sigma$ , it follows that also  $\mu' \subseteq \sigma'$  and  $\mu \subseteq \sigma$ . Due to  $s > 0$ , we have that  $\sigma = \bigcup_{B \subseteq \sigma, |B| \leq s} B \subseteq \bigcup_{B \subseteq \sigma, |B| \leq s} \varrho_B = \mu$ , wherefore  $\mu = \sigma$ . This shows that  $(\sigma, \sigma')$  has the right form to fit into the set on the right-hand side, provided we establish that the non-empty set  $\mathcal{T}$  is  $s$ -directed.

For this goal, we consider  $t \leq s$  subsets  $B_0, \dots, B_{t-1} \subseteq \sigma$  subject to the condition  $|B_i| \leq s$  for each  $0 \leq i < t$  and tuples  $r_i \in \varrho_{B_i} \subseteq \sigma$ . Let us define  $C := \{r_i \mid 0 \leq i < t\} \subseteq \sigma$ . As  $|C| \leq s$ , the pair  $(\varrho_C, \varrho'_C)$  belongs to  $\mathcal{T}$  by definition. Thus  $C \subseteq \varrho_C$  demonstrates  $s$ -directedness, concluding the proof.  $\square$

**Remark 5.18.** The inclusion “ $\subseteq$ ” in Proposition 5.17 fails to hold for  $s = 0$ . Consider, for example, any pair of relations  $\varrho' \subseteq \varrho \subsetneq A^m$  for some fixed  $m \in \mathbb{N}$ . Define  $Q := \rightarrow \{(\varrho, \varrho')\}^\leftarrow = \left\{(\sigma, \sigma') \in \text{Rp}_A^{(m)} \mid \varrho' \subseteq \sigma' \subseteq \sigma \subseteq \varrho\right\}$ , then  $Q$  is certainly closed w.r.t. relaxation, and, moreover, it is not hard to see that it is also closed under arbitrary non-empty unions, i.e. 0-directed unions. Therefore, the set appearing on the right-hand side in Proposition 5.17 is contained in  $Q$ . Now, the set  $0\text{-LOC}_A Q = \left\{(\sigma, \sigma') \in \text{Rp}_A^{(m)} \mid \exists (\mu, \mu') \in Q^{(m)} : \sigma' \supseteq \mu'\right\}$  clearly contains  $(A^m, \varrho')$ , but this pair does not belong to the set on the right for it fails to belong to  $Q$  due to  $A^m \not\subseteq \varrho$ .

In generalisation of Proposition 1.13(i) of [29, p. 18] (see also [28, Proposition 1.6(i), p. 256]), a similar characterisation as above can be achieved for the local closure operator.

**Corollary 5.19.** *For any set  $Q \subseteq \text{Rp}_A$  of relation pairs, we have*

$$\text{LOC}_A Q = \bigcup_{m \in \mathbb{N}} \left\{ (\sigma, \sigma') \in \text{Rp}_A^{(m)} \mid \begin{array}{l} \exists \sigma'' \in \text{R}_A^{(m)} : \sigma'' \subseteq \sigma' \wedge \\ \exists \mathcal{T} \subseteq \text{Rp}_A^{(m)} \text{ } \aleph_0\text{-directed:} \\ (\sigma, \sigma'') = \bigcup \mathcal{T} \wedge \\ \forall (\varrho, \varrho') \in \mathcal{T} \exists \tilde{\varrho} \in \text{R}_A^{(m)} : \\ \varrho \subseteq \tilde{\varrho} \wedge (\tilde{\varrho}, \varrho') \in Q^{(m)} \end{array} \right\}.$$

*Proof.* The proof of Proposition 5.17 can be literally copied employing the following modifications: the use of Lemma 2.14 has to be substituted by Corollary 2.15; every occurrence of “ $s$ -directed”, “ $s$ -locally closed” and the operator

$s\text{-LOC}_A$  has to be replaced by “ $\aleph_0$ -directed”, “locally closed” and the operator  $\text{LOC}_A$ , respectively; every restriction of the form  $|B| \leq s$ ,  $|B_i| \leq s$  and  $|C| \leq s$  should be changed to  $|B| < \aleph_0$ ,  $|B_i| \leq \aleph_0$  and  $|C| < \aleph_0$ , respectively; and, finally, the phrase “Due to  $s > 0$ ,” is to be removed completely.  $\square$

Placing an additional closure requirement on the sets  $Q \subseteq \text{Rp}_A$  in Corollary 5.19, we can sharpen the statement by replacing  $\aleph_0$ -directed unions by directed unions.

**Corollary 5.20.** *For any set  $Q \subseteq \text{Rp}_A$  of relation pairs that is closed under arbitrary intersections of pairs of identical arity, the following equality holds:*

$$\text{LOC}_A Q = \bigcup_{m \in \mathbb{N}} \left\{ (\sigma, \sigma') \in \text{Rp}_A^{(m)} \mid \begin{array}{l} \exists \sigma'' \in \text{Rp}_A^{(m)} : \sigma'' \subseteq \sigma' \wedge \\ \exists \mathcal{T} \subseteq \text{Rp}_A^{(m)} \text{ directed:} \\ (\sigma, \sigma'') = \bigcup \mathcal{T} \wedge \\ \forall (\varrho, \varrho') \in \mathcal{T} \exists \tilde{\varrho} \in \text{Rp}_A^{(m)} : \\ \varrho \subseteq \tilde{\varrho} \wedge (\tilde{\varrho}, \varrho') \in Q^{(m)} \end{array} \right\}.$$

*Proof.* The proof of the inclusion “ $\supseteq$ ” stays the same as in Corollary 5.19, one just reads “directed” in place of “ $\aleph_0$ -directed”.

The dual inclusion requires a few more changes. Consider any relation pair  $(\sigma, \sigma') \in \text{LOC}_A^{(m)} Q$ ,  $m \in \mathbb{N}$ . Then for any finite subset  $B \subseteq \sigma$ , the set  $\Sigma_B := \{(\varrho, \varrho') \in Q^{(m)} \mid B \subseteq \varrho \wedge \varrho' \subseteq \sigma'\}$  is non-empty. Let us define the pair  $(\tilde{\varrho}_B, \varrho'_B) := \bigcap \Sigma_B$ , that is, we have  $\tilde{\varrho}_B = \bigcap_{(\varrho, \varrho') \in \Sigma_B} \varrho$  and  $\varrho'_B = \bigcap_{(\varrho, \varrho') \in \Sigma_B} \varrho'$ . Since  $\Sigma_B \neq \emptyset$ , we have  $\varrho'_B \subseteq \varrho'$  for some  $(\varrho, \varrho') \in Q^{(m)}$ , i.e.  $\varrho'_B \subseteq \varrho' \subseteq \sigma' \subseteq \sigma$ . Thus,  $\varrho'_B \subseteq \tilde{\varrho}_B \cap \sigma =: \varrho_B$ . Moreover, we know that  $B \subseteq \varrho_B \subseteq \tilde{\varrho}_B$  since  $B \subseteq \varrho$  for any  $(\varrho, \varrho') \in \Sigma_B$ . By closure w.r.t. intersection, we obtain  $(\tilde{\varrho}_B, \varrho'_B) \in Q^{(m)}$ , and so  $(\tilde{\varrho}_B, \varrho'_B) \in \Sigma_B \subseteq Q^{(m)}$ . Defining  $\mathcal{T} := \{(\varrho_B, \varrho'_B) \mid B \subseteq \sigma \wedge |B| < \aleph_0\}$ , we can continue as in the proof of Corollary 5.19; only the final paragraph needs further modifications to demonstrate that  $\mathcal{T}$  is directed and not only  $\aleph_0$ -directed.

For this we consider any finite subset  $\mathcal{B} \subseteq \{B \subseteq \sigma \mid |B| < \aleph_0\}$ . Clearly, the union  $C := \bigcup \mathcal{B}$  is again a finite subset of  $\sigma$ . Moreover, for all  $B \in \mathcal{B}$ , we have  $B \subseteq C$  and hence  $\Sigma_C \subseteq \Sigma_B$ , which implies  $\tilde{\varrho}_B \subseteq \tilde{\varrho}_C$  and, consequently,  $\varrho_B = \sigma \cap \tilde{\varrho}_B \subseteq \sigma \cap \tilde{\varrho}_C = \varrho_C$ . Since this holds for all  $B \in \mathcal{B}$ , we get  $\bigcup_{B \in \mathcal{B}} \varrho_B \subseteq \varrho_C$ , proving directedness of  $\mathcal{T}$ .  $\square$

## 6. SPECIAL CASES

**6.1. Proper semiclones.** Based on the results of the previous section, we may also characterise all  $s$ -locally closed semiclones that fail to be clones.

**Proposition 6.1.** *For any parameter  $s \in \mathbb{N}$  and any carrier set  $A$ , the collection  $\{F \in \mathcal{S}_A \setminus \mathcal{L}_A \mid F \text{ } s\text{-locally closed}\}$  can be relationally described in the form  $\left\{ \text{Polp}_A Q \mid Q \subseteq \text{Rp}_A^{(\leq s)} \wedge \exists (\varrho, \varrho') \in Q : \varrho' \subsetneq \varrho \right\}$ .*

*Proof.* Consider a set  $Q \subseteq \text{Rp}_A^{(\leq s)}$  such that  $\varrho' \subsetneq \varrho$  holds for at least one relation pair  $(\varrho, \varrho') \in Q$ . By Lemma 3.7 we have  $\text{Polp}_A Q \in \mathcal{S}_A$ , and moreover, this set is  $s$ -locally closed by Lemma 3.10. Further, Lemma 3.9 ensures that  $\text{Polp}_A Q \notin \mathcal{L}_A$ .

Conversely, if  $F \in \mathcal{S}_A \setminus \mathcal{L}_A$  is  $s$ -locally closed, then by Corollary 5.7, we have  $F = s\text{-Loc}_A[F]_{\mathcal{O}_A} = \text{Polp}_A Q$  where  $Q = \text{Invp}_A^{(\leq s)} F$  (cf. Theorem 5.3). If we had  $\varrho = \varrho'$  for all  $(\varrho, \varrho') \in Q$ , then it would follow  $\text{id}_A \in \text{Polp}_A Q = F$ , implying  $F \in \mathcal{L}_A$  according to Lemma 3.5(d). Hence, there is at least one  $(\varrho, \varrho') \in Q$  where  $\varrho' \subsetneq \varrho$ .  $\square$

In a very analogous fashion we may prove the following result.

**Proposition 6.2.** *For all carriers  $A$  we have the equality*

$$\begin{aligned} \{F \in \mathcal{S}_A \setminus \mathcal{L}_A \mid F \text{ locally closed}\} \\ = \{\text{Polp}_A Q \mid Q \subseteq \text{Rp}_A \wedge \exists (\varrho, \varrho') \in Q: \varrho' \subsetneq \varrho\}. \end{aligned}$$

*Proof.* In the proof of Proposition 6.1 replace the use of Lemma 3.10 by Corollary 3.11, Theorem 5.3 by Corollary 5.6, the operators  $s\text{-Loc}_A$  by  $\text{Loc}_A$ ,  $\text{Invp}_A^{(\leq s)}$  by  $\text{Invp}_A$  and  $\text{Rp}_A^{(\leq s)}$  by  $\text{Rp}_A$ , respectively, and “ $s$ -locally” by “locally”.  $\square$

Using the theory of the previous sections, we can also prove a decidability result regarding the question if a clone with projections removed yields a semicclone, or if the non-trivial functions generate the projections.

For this we need a more detailed analysis of the process generating  $\Gamma_F(B)$  occurring in Lemma 5.1.

**Lemma 6.3.** *Let  $K$  be any set,  $B \subseteq A^K$  and  $F \subseteq \mathcal{O}_A$ . Define  $R_0 := B$  and*

$$\begin{aligned} S_j &:= \bigcup_{n \in \mathbb{N}} \left\{ f \circ (g_0, \dots, g_{n-1}) \mid f \in F^{(n)} \wedge (g_0, \dots, g_{n-1}) \in R_j^n \right\} \\ R_{j+1} &:= R_j \cup S_j \end{aligned}$$

for  $j \in \mathbb{N}$ ; set  $R := \bigcup_{j \in \mathbb{N}} R_j$  and  $S := \bigcup_{j \in \mathbb{N}} S_j$ . Employing the straightforward generalisation of the preservation concept to infinite arities, the pair  $(R, S)$  is the least (w.r.t.  $\leq$ ) member of the set

$$Q_B := \{(\varrho, \varrho') \mid B \cup \varrho' \subseteq \varrho \subseteq A^K, (\varrho, \varrho') \text{ is preserved by all } f \in F\}.$$

If  $R_n = R_{n+1}$ , i.e.  $S_n \subseteq R_n$ , holds for some  $n \in \mathbb{N}$ , then it is  $S_m = S_n$  and  $R_m = R_n$  for all  $m \geq n$ . Therefore, for finite  $A$  and finite  $K$ , the condition  $R_n = R_{n+1}$  is satisfied for some  $n \leq |A^K|$ .

*Proof.* First, we note that, by definition,  $R_j \subseteq R_{j+1}$ , which implies  $S_j \subseteq S_{j+1}$ , holds for all  $j \in \mathbb{N}$ . Hence, the unions defining  $R$  and  $S$  are directed.

It is not difficult to see that  $(R, S)$  belongs to  $Q_B$ . Namely, we have  $B = R_0 \subseteq R$  and  $S_j \subseteq R_{j+1} \subseteq R$  for every  $j \in \mathbb{N}$ , whence  $S \subseteq R$ . To prove that  $(R, S)$  is preserved by every  $n$ -ary  $f \in F$ , one considers an  $n$ -tuple of tuples  $(g_0, \dots, g_{n-1}) \in R^n$ . Due to directedness of the union producing  $R$  and finiteness of  $n$ , there exists one  $j \in \mathbb{N}$  such that  $(g_0, \dots, g_{n-1}) \in R_j^n$ , wherefore  $f \circ (g_0, \dots, g_{n-1}) \in S_j \subseteq S$ . Consequently,  $(R, S)$  is preserved by every member of  $F$  and thus belongs to  $Q_B$ .

Second, take any pair  $(\varrho, \varrho') \in Q_B$ . By definition, we have  $R_0 = B \subseteq \varrho$ . Moreover, supposing that  $R_j \subseteq \varrho$ , by the preservation condition, we get that  $S_j \subseteq \varrho' \subseteq \varrho$  and hence  $R_{j+1} = R_j \cup S_j \subseteq \varrho$ , as well as  $S_j \subseteq \varrho'$ . Thus, by induction we have shown  $R = \bigcup_{j \in \mathbb{N}} R_j \subseteq \varrho$  and  $S = \bigcup_{j \in \mathbb{N}} S_j \subseteq \varrho'$ . This proves that  $(R, S) \leq (\varrho, \varrho')$ , whence  $(R, S)$  is the  $\leq$ -least member of  $Q_B$ .

It is easy to check by induction that  $R_n = R_{n+1}$  entails  $S_m = S_n$  and  $R_m = R_n$  for all  $m \geq n$ . Moreover, if  $R_0, R_1, \dots, R_n$  are pairwise distinct (i.e. form a strictly increasing chain  $R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_n$ ), then  $n \leq |R_n| \leq |A^K|$ . Therefore, for finite  $A$  and  $K$ , the condition  $R_n = R_{n+1}$  must be satisfied for some  $n \leq |A^K|$ .  $\square$

The following lemma goes back to an idea by Peter Mayr.

**Lemma 6.4.** *For  $F \subseteq O_A$  we have  $[F \setminus J_A]_{O_A} = [\langle F \rangle_{O_A} \setminus J_A]_{O_A}$ .*

*Proof.* Using Lemma 3.5(b) we have

$$\begin{aligned} F \setminus J_A &\subseteq \langle F \rangle_{O_A} \setminus J_A = \langle F \setminus J_A \rangle_{O_A} \setminus J_A = ([F \setminus J_A]_{O_A} \cup J_A) \setminus J_A \\ &= [F \setminus J_A]_{O_A} \setminus J_A \subseteq [F \setminus J_A]_{O_A}, \end{aligned}$$

whence we obtain  $[F \setminus J_A]_{O_A} = [\langle F \rangle_{O_A} \setminus J_A]_{O_A}$  by another application of the operator  $[]_{O_A}$ .  $\square$

With this result we can now prove the problem of whether a finitely generated clone on a finite set generates projections from its non-trivial members to be decidable.

**Proposition 6.5.** *For both  $A$  and  $F \subseteq O_A$  finite, it is decidable whether  $\langle F \rangle_{O_A} \setminus J_A \in \mathcal{S}_A$ .*

*Proof.* Since  $\langle F \rangle_{O_A}$  is a clone, and thus, in particular, a semiclone, containing  $\langle F \rangle_{O_A} \setminus J_A$ , we have  $\langle F \rangle_{O_A} \setminus J_A \subseteq [\langle F \rangle_{O_A} \setminus J_A]_{O_A} \subseteq \langle F \rangle_{O_A}$ . Therefore, by Lemma 3.5(c), the conditions  $\langle F \rangle_{O_A} \setminus J_A \notin \mathcal{S}_A$ ,  $\langle F \rangle_{O_A} \setminus J_A \subsetneq [\langle F \rangle_{O_A} \setminus J_A]_{O_A}$ ,  $[\langle F \rangle_{O_A} \setminus J_A]_{O_A} \cap J_A \neq \emptyset$ ,  $J_A \subseteq [\langle F \rangle_{O_A} \setminus J_A]_{O_A}$  and  $\text{id}_A \in [\langle F \rangle_{O_A} \setminus J_A]_{O_A}^{(1)}$  are all equivalent. By Lemma 6.4 we get  $[\langle F \rangle_{O_A} \setminus J_A]_{O_A}^{(1)} = [F \setminus J_A]_{O_A}^{(1)}$ , and using Corollary 5.2 for  $n = 1$ ,  $X = A$  and some bijection  $\beta$  between  $A$  and its cardinality, we have a description of the invariant pair  $(\varrho_{A,1}, \varrho'_{A,1}) = \Gamma_{F \setminus J_A}(\{\text{id}_A \circ \beta\})$ . Via Lemma 6.3 this invariant relation pair generated by  $\text{id}_A \circ \beta$  can be expressed as  $(\bigcup_{j \in \mathbb{N}} R_j, \bigcup_{j \in \mathbb{N}} S_j)$  and finiteness of  $A$  guarantees that  $R_n = R_{n+1}$  happens for some  $n \leq |A^A|$ . This implies that  $\left\{ f \circ \beta \mid f \in [F \setminus J_A]_{O_A}^{(1)} \right\} = \varrho'_{A,1}$  can be written as the finite union  $\bigcup_{0 \leq j \leq |A^A|} S_j$ , which due to finiteness of  $F$  can be straightforwardly calculated using the definitions of Lemma 6.3. Hence, one may check if  $\text{id}_A$  belongs to  $[F \setminus J_A]_{O_A}^{(1)}$  by checking if  $\beta = \text{id}_A \circ \beta$  belongs to this union.  $\square$

**6.2. Closed transformation semigroups.** By considering just unary parts we can obtain characterisations of locally and  $s$ -locally closed (proper) transformation semigroups, respectively.

**Proposition 6.6.** *For  $s \in \mathbb{N}$  and a set  $H \subseteq O_A^{(1)}$  of transformations the following facts are equivalent*

- (a)  $H$  is an  $s$ -locally closed transformation semigroup [and  $\text{id}_A \notin H$ ].
- (b)  $H = \text{Polp}_A^{(1)} \text{Invp}_A^{(\leq s)} H$  [and there is  $(\varrho, \varrho') \in \text{Invp}_A^{(\leq s)} H$  such that  $\varrho' \neq \varrho$ ].
- (c)  $H = \text{Polp}_A^{(1)} Q$  for some set  $Q \subseteq \text{Rp}_A^{(\leq s)}$  [where  $\varrho \neq \varrho'$  for some  $(\varrho, \varrho') \in Q$ ].



For non-empty carrier sets  $A$ , the arity restrictions “ $\leq s$ ” can be replaced by simply “ $s$ ”.

*Proof.* We shall prove the implications “(a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (a)”.

“(a)  $\Rightarrow$  (b)” Let  $S := [H]_{O_A}$  and  $T := s\text{-Loc}_A S$ . Using Lemma 3.2 we obtain  $H = S^{(1)}$ . We also have  $T = s\text{-Loc}_A S = s\text{-Loc}_A [H]_{O_A} = \text{Polp}_A \text{Invp}_A^{(\leq s)} H$  by Theorem 5.3, whence  $\text{Polp}_A^{(1)} \text{Invp}_A^{(\leq s)} H = s\text{-Loc}_A^{(1)} S = s\text{-Loc}_A (S^{(1)})$  is equal to  $s\text{-Loc}_A H = H$  due to  $H$  being  $s$ -locally closed. For the alternative reading note that  $\text{id}_A$  would belong to  $\text{Polp}_A^{(1)} \text{Invp}_A^{(\leq s)} H = H$  if all pairs in  $\text{Invp}_A^{(\leq s)} H$  had equal components. Furthermore, if  $A \neq \emptyset$ , then one may use Corollary 5.5 instead of Theorem 5.3 to change “ $\leq s$ ” into “ $s$ ”.

“(b)  $\Rightarrow$  (c)” Simply choose  $Q := \text{Invp}_A^{(\leq s)} H$ .

“(c)  $\Rightarrow$  (a)” Clearly,  $\text{Polp}_A Q \in \mathcal{S}_A$  by Lemma 3.7, so Corollary 3.3 yields that  $H = \text{Polp}_A^{(1)} Q$  is a transformation semigroup. Since  $Q \subseteq \text{Rp}_A^{(\leq s)}$ , we obtain  $\text{Polp}_A Q = s\text{-Loc}_A \text{Polp}_A Q$  by Lemma 3.10, wherefore we may express  $H$  as  $H = \text{Polp}_A^{(1)} Q = s\text{-Loc}_A^{(1)} \text{Polp}_A Q = s\text{-Loc}_A \text{Polp}_A^{(1)} Q = s\text{-Loc}_A H$ . Note for the second reading that  $\text{id}_A$  does not preserve relation pairs  $\varrho' \subsetneq \varrho$ .  $\square$

In an analogous way we may characterise the locally closed (proper) transformation semigroups.

**Proposition 6.7.** *For any set  $H \subseteq O_A^{(1)}$  of transformations the following facts are equivalent:*

- (a)  $H$  is a locally closed transformation semigroup [and  $\text{id}_A \notin H$ ].
- (b)  $H = \text{Polp}_A^{(1)} \text{Invp}_A H$  [and there is  $(\varrho, \varrho') \in \text{Invp}_A H$  such that  $\varrho' \neq \varrho$ ].
- (c)  $H = \text{Polp}_A^{(1)} Q$  for some set  $Q \subseteq \text{Rp}_A$  [where  $\varrho \neq \varrho'$  for some  $(\varrho, \varrho') \in Q$ ].

*Proof.* In the proof of Proposition 6.6 substitute “ $s$ -locally” by “locally”, the operators  $\text{Invp}_A^{(\leq s)}$  by  $\text{Invp}_A$ ,  $s\text{-Loc}_A$  by  $\text{Loc}_A$ , and  $\text{Rp}_A^{(\leq s)}$  by  $\text{Rp}_A$ , respectively, and the applications of Theorem 5.3 by Corollary 5.6 and of Lemma 3.10 by Corollary 3.11.  $\square$

By intersecting (in a similar way as outlined in this subsection) with other classes of functions, for example, the set of all permutations instead of all unary operations, one can obtain further characterisations of locally closed classes of functions in terms of relation pairs. Continuing the example of permutations, one may get a characterisation of all locally [ $s$ -locally] closed (proper) transformation semigroups that consist of permutations only. As on finite carrier sets every permutation has a finite order, such a result is necessarily more appealing on infinite domains.

**6.3. Classical Pol-Inv Galois correspondence.** Here we demonstrate that it is not difficult to derive the characterisations of the closure operators of the GALOIS connection given by polymorphisms and invariant relations from our theorems above. In this respect, we first consider the framework including nullary operations and relations as discussed in [3]; from there it will be a small step to obtain the variants known from [29, 28].

First we recollect information concerning the relationship of the operators  $\text{Pol}_A$  and  $\text{Inv}_A$  w.r.t.  $\text{Polp}_A$  and  $\text{Invp}_A$ , which we already briefly discussed before Lemma 2.3.

**Lemma 6.8.** *For  $Q \subseteq R_A$ ,  $F \subseteq O_A$  and any  $n \in \mathbb{N}$  we have*

$$\begin{aligned} \text{Pol}_A^{(n)} Q &= \text{Polp}_A^{(n)} \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\}, \\ \text{Inv}_A^{(n)} F &= \left\{ \varrho \mid (\varrho, \varrho) \in \text{Invp}_A^{(n)} F \right\}. \end{aligned}$$

*Proof.* The claim follows since a function  $f \in O_A$  preserves a relation  $\varrho \in R_A$  (w.r.t. Pol-Inv) if and only if  $f \triangleright (\varrho, \varrho)$  (in the sense of Polp-Invp).  $\square$

We shall also need to express the set  $\text{Invp}_A F$  in terms of  $\text{Inv}_A F$  for  $F \subseteq O_A$  containing at least one projection.

**Lemma 6.9.** *Suppose, a set  $F \subseteq O_A$  of operations satisfies  $F \cap J_A \neq \emptyset$ , then  $\text{Invp}_A F = \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in \text{Inv}_A^{(m)} F \right\}$ .*

*Proof.* Put  $Q := \text{Invp}_A F$ . By Lemma 3.7,  $\text{Polp}_A Q$  is a semiclone, and, as  $\text{Polp}_A Q \supseteq F$  contains projections, it is even a clone (cp. Lemma 3.5(d)). Hence, Lemma 3.9 yields that  $Q \subseteq \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in R_A^{(m)} \right\}$ , which implies  $Q = \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid (\varrho, \varrho) \in \text{Invp}_A^{(m)} F \right\} = \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in \text{Inv}_A^{(m)} F \right\}$  (cf. Lemma 6.8 for the second equality).  $\square$

This enables us now to derive the characterisation of the closure operators  $\text{Pol}_A \text{Inv}_A^{(\leq s)}$  and  $\text{Pol}_A \text{Inv}_A^{(s)}$ .

**Theorem 6.10.** *We have  $s\text{-Loc}_A \langle F \rangle_{O_A} = \text{Pol}_A \text{Inv}_A^{(\leq s)} F$  for  $F \subseteq O_A$  and  $s \in \mathbb{N}$ ; moreover, the equality  $s\text{-Loc}_A \langle F \rangle_{O_A} = \text{Pol}_A \text{Inv}_A^{(s)} F$  holds.*

*Proof.* Using Lemma 3.5(b) we can write  $\langle F \rangle_{O_A} = [F \cup \{\text{id}_A\}]_{O_A}$ , hence Theorem 5.3 entails that  $s\text{-Loc}_A \langle F \rangle_{O_A} = s\text{-Loc}_A [F \cup \{\text{id}_A\}]_{O_A}$  coincides with  $\text{Polp}_A \text{Invp}_A^{(\leq s)} (F \cup \{\text{id}_A\})$ , which by Lemma 6.9 equals

$$\begin{aligned} & \text{Polp}_A \biguplus_{m=0}^s \left\{ (\varrho, \varrho) \mid \varrho \in \text{Inv}_A^{(m)} (F \cup \{\text{id}_A\}) \right\} \\ &= \text{Polp}_A \biguplus_{m=0}^s \left\{ (\varrho, \varrho) \mid \varrho \in \text{Inv}_A^{(m)} F \right\} \quad (\text{since } \text{Inv}_A (F \cup \{\text{id}_A\}) = \text{Inv}_A F) \\ &= \bigcap_{m=0}^s \text{Polp}_A \left\{ (\varrho, \varrho) \mid \varrho \in \text{Inv}_A^{(m)} F \right\} \\ &= \bigcap_{m=0}^s \text{Pol}_A \text{Inv}_A^{(m)} F = \text{Pol}_A \text{Inv}_A^{(\leq s)} F. \end{aligned} \quad (\text{cf. Lemma 6.8})$$

For  $A \neq \emptyset$  we may replace Theorem 5.3 by Corollary 5.5 and therefore the operator  $\text{Polp}_A \text{Invp}_A^{(\leq s)}$  by  $\text{Polp}_A \text{Invp}_A^{(s)}$ . The rest of the argument is analogous to the above. For  $A = \emptyset$ ,  $\text{Pol}_A \text{Inv}_A^{(s)} F$  and  $\text{Pol}_A \text{Inv}_A^{(\leq s)} = s\text{-Loc}_A \langle F \rangle_{O_A}$  are both clones on  $\emptyset$ , but there exists only one such structure, namely  $O_\emptyset$ .  $\square$

As a consequence, we get Theorem 3.17 from [3, p. 29]:

**Corollary 6.11.** *For  $F \subseteq O_A$  we have  $\text{Loc}_A \langle F \rangle_{O_A} = \text{Pol}_A \text{Inv}_A F$ .*

*Proof.* By definition of the operator  $\text{Loc}_A$  we have

$$\begin{aligned} \text{Loc}_A \langle F \rangle_{O_A} &= \bigcap_{s \in \mathbb{N}} s\text{-Loc}_A \langle F \rangle_{O_A} \stackrel{6.10}{=} \bigcap_{s \in \mathbb{N}} \text{Pol}_A \text{Inv}_A^{(\leq s)} F \\ &= \text{Pol}_A \bigcup_{s \in \mathbb{N}} \text{Inv}_A^{(\leq s)} F = \text{Pol}_A \text{Inv}_A F. \quad \square \end{aligned}$$

In contrast to semiclones, nullary relations are never needed to discern locally closed clones. Even more generally, invariants of small arity may always be neglected.

**Corollary 6.12.** *For a set of operations  $F \subseteq O_A$  and any arity  $m \in \mathbb{N}$  we have the equality  $\text{Pol}_A \text{Inv}_A^{(\geq m)} F = \text{Pol}_A \text{Inv}_A F = \text{Loc}_A \langle F \rangle_{O_A}$ .*

*Proof.* As a consequence of Theorem 6.10 we have

$$\begin{aligned} \text{Pol}_A \text{Inv}_A F &= \bigcap_{s \in \mathbb{N}} \text{Pol}_A \text{Inv}_A^{(s)} F = \bigcap_{s \in \mathbb{N}} s\text{-Loc}_A \langle F \rangle_{O_A} \\ &= \bigcap_{\substack{s \in \mathbb{N} \\ s \geq m}} s\text{-Loc}_A \langle F \rangle_{O_A} = \bigcap_{\substack{s \in \mathbb{N} \\ s \geq m}} \text{Pol}_A \text{Inv}_A^{(s)} F = \text{Pol}_A \text{Inv}_A^{(\geq m)} F \end{aligned}$$

for any  $F \subseteq O_A$ .  $\square$

The following observation will be helpful in deriving the original formulations (without nullary operations) of the previously presented results.

**Lemma 6.13.** *For operations  $F \subseteq O_A \setminus O_A^{(0)}$  of positive arity and every  $s \in \mathbb{N}$  the equality  $s\text{-Loc}_A^{(0)} \langle F \rangle_{O_A} = \text{Loc}_A^{(0)} \langle F \rangle_{O_A} = \langle F \rangle_{O_A}^{(0)} = \emptyset$  holds.*

*Proof.* Since  $F \subseteq O_A \setminus O_A^{(0)}$ , which is a clone, we obtain  $\langle F \rangle_{O_A} \subseteq O_A \setminus O_A^{(0)}$ , i.e.  $\langle F \rangle_{O_A}^{(0)} = \emptyset$  and thus  $\text{Loc}_A^{(0)} \langle F \rangle_{O_A} \subseteq s\text{-Loc}_A^{(0)} \langle F \rangle_{O_A} = s\text{-Loc}_A \langle F \rangle_{O_A}^{(0)} = \emptyset$ .  $\square$

**Corollary 6.14.** *Let  $F \subseteq O_A \setminus O_A^{(0)}$  be without nullary operations and  $s \in \mathbb{N}$ , then we have the equalities  $\text{Pol}_A^{(>0)} \text{Inv}_A F = \text{Pol}_A^{(>0)} \text{Inv}_A^{(>0)} F = \text{Loc}_A \langle F \rangle_{O_A}$  and  $\text{Pol}_A^{(>0)} \text{Inv}_A^{(s)} F = s\text{-Loc}_A \langle F \rangle_{O_A}$ .*

*Proof.* Combining Lemma 6.13 with Corollary 6.12 (for  $m = 1$ ) yields emptiness of the set  $\text{Pol}_A^{(0)} \text{Inv}_A^{(>0)} F = \text{Pol}_A^{(0)} \text{Inv}_A F = \text{Loc}_A^{(0)} \langle F \rangle_{O_A}$ . Therefore,

$$\begin{aligned} \text{Pol}_A^{(>0)} \text{Inv}_A F &= \text{Pol}_A \text{Inv}_A F = \text{Loc}_A \langle F \rangle_{O_A} \\ &= \text{Pol}_A \text{Inv}_A^{(>0)} F = \text{Pol}_A^{(>0)} \text{Inv}_A^{(>0)} F. \end{aligned}$$

In a similar way, we may invoke Theorem 6.10 together with Lemma 6.13 to get  $\text{Pol}_A^{(0)} \text{Inv}_A^{(s)} F = s\text{-Loc}_A^{(0)} \langle F \rangle_{O_A} = \emptyset$ . Using again Theorem 6.10 we can infer  $\text{Pol}_A^{(>0)} \text{Inv}_A^{(s)} F = \text{Pol}_A \text{Inv}_A^{(s)} F = s\text{-Loc}_A \langle F \rangle_{O_A}$ .  $\square$

The equalities  $\text{Loc}_A \langle F \rangle_{O_A} = \text{Pol}_A^{(>0)} \text{Inv}_A^{(>0)} F$  for  $F \subseteq O_A$  without nullary operations, as well as  $s\text{-Loc}_A \langle F \rangle_{O_A} = \text{Pol}_A^{(>0)} \text{Inv}_A^{(s)} F$  whenever  $s > 0$ , express two of the main results regarding Pol-Inv that one finds in [28, Theorem 3.2, p. 260], [29, Theorem 4.1, p. 31], where neither nullary operations nor nullary relations were considered.

In order to attack the relational side of the Pol-Inv GALOIS correspondence, we need to express generated relational clones, i.e. the closure  $[\cdot]_{R_A}$  of a set of relations under general superpositions, in terms of generated relation pair clones. This is prepared in the following lemma.

**Lemma 6.15.** *For any set  $Q \subseteq R_A$  we have*

$$\biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in [Q]_{R_A}^{(m)} \right\} = \left[ \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\} \right]_{R_{p_A}}.$$

*Proof.* Since  $Q \subseteq [Q]_{R_A}$ , the set  $P := \biguplus_{m \in \mathbb{N}} \{(\varrho, \varrho) \mid \varrho \in Q^{(m)}\}$  obviously is a subset of  $\biguplus_{m \in \mathbb{N}} \{(\varrho, \varrho) \mid \varrho \in [Q]_{R_A}^{(m)}\}$ . By Lemma 4.4, the latter set is a relation pair clone, whence it includes  $[P]_{R_{p_A}}$ .

Conversely, the set  $\left\{ \sigma \mid (\sigma, \sigma) \in [P]_{R_{p_A}} \right\}$ , which contains  $Q$ , is a relational clone by Lemma 4.5; hence  $[Q]_{R_A} \subseteq \left\{ \sigma \mid (\sigma, \sigma) \in [P]_{R_{p_A}} \right\}$ . Thus, whenever  $\varrho \in [Q]_{R_A}^{(m)}$  for  $m \in \mathbb{N}$ , we find  $(\varrho, \varrho)$  in  $[P]_{R_{p_A}}$ , which proves “ $\subseteq$ ”.  $\square$

Similarly, we have to relate the  $s$ -local closures of sets of relations (cf. page 8) and that of sets of relation pairs.

**Lemma 6.16.** *For any set  $Q \subseteq R_A$  and  $s \in \mathbb{N}_+$  we have*

$$s\text{-LOC}_A \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\} = \biguplus_{m \in \mathbb{N}} \left\{ (\sigma, \sigma) \mid \sigma \in s\text{-LOC}_A^{(m)} Q \right\}.$$

*Proof.* It is sufficient to prove for fixed  $m \in \mathbb{N}$  that the  $m$ -ary part of the left set,  $s\text{-LOC}_A^{(m)} \biguplus_{n \in \mathbb{N}} \{(\varrho, \varrho) \mid \varrho \in Q^{(n)}\} = s\text{-LOC}_A \{(\varrho, \varrho) \mid \varrho \in Q^{(m)}\}$ , coincides with  $\left\{ (\sigma, \sigma) \mid \sigma \in s\text{-LOC}_A^{(m)} Q \right\}$ . By definition of the  $s$ -local closure we have that  $s\text{-LOC}_A \{(\varrho, \varrho) \mid \varrho \in Q^{(m)}\}$  equals

$$\left\{ (\sigma, \sigma') \in R_{p_A}^{(m)} \mid \forall B \subseteq \sigma, |B| \leq s \exists \varrho \in Q^{(m)} : B \subseteq \varrho \wedge \varrho \subseteq \sigma' \right\}.$$

Due to  $s > 0$  any relation pair  $(\sigma, \sigma')$  belonging to the previously displayed set satisfies  $\sigma = \bigcup \{B \subseteq \sigma \mid |B| \leq s\} \subseteq \sigma' \subseteq \sigma$ , i.e.  $\sigma = \sigma'$ . Therefore, we obtain

$$\begin{aligned} & s\text{-LOC}_A \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\} \\ &= \left\{ (\sigma, \sigma) \mid \sigma \in R_A^{(m)} \wedge \forall B \subseteq \sigma, |B| \leq s \exists \varrho \in Q^{(m)} : B \subseteq \varrho \subseteq \sigma \right\} \\ &= \left\{ (\sigma, \sigma) \mid \sigma \in R_A^{(m)} \wedge \sigma \in s\text{-LOC}_A Q \right\} \end{aligned}$$

as desired.  $\square$

We may now characterise the closure operator  $\text{Inv}_A \text{Pol}_A^{(\leq s)}$  in terms of the  $s$ -local closure and the generated relational clone.

**Theorem 6.17.** *For any parameter  $s \in \mathbb{N}$  and any set of relations  $Q \subseteq R_A$  the equalities  $s\text{-LOC}_A [Q]_{R_A} = \text{Inv}_A \text{Pol}_A^{(\leq s)} Q = \text{Inv}_A \text{Pol}_A^{(0, s)} Q$  hold.*

*Proof.* Using the previous results we may calculate for  $Q \subseteq R_A$  and  $s \in \mathbb{N}_+$

$$\text{Inv}_A \text{Pol}_A^{(\leq s)} Q \stackrel{6.8}{=} \text{Inv}_A \text{Polp}_A^{(\leq s)} \biguplus_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\}$$

$$\begin{aligned}
&\stackrel{6.8}{=} \left\{ \sigma \mid (\sigma, \sigma) \in \text{Inv}_A \text{Pol}_A^{(\leq s)} \bigcup_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\} \right\} \\
&\stackrel{5.10}{=} \left\{ \sigma \mid (\sigma, \sigma) \in s\text{-LOC}_A \left[ \bigcup_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in Q^{(m)} \right\} \right]_{\text{Rp}_A} \right\} \\
&\stackrel{6.15}{=} \left\{ \sigma \mid (\sigma, \sigma) \in s\text{-LOC}_A \bigcup_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in [Q]_{\text{R}_A}^{(m)} \right\} \right\} \\
&\stackrel{6.16}{=} \left\{ \sigma \mid (\sigma, \sigma) \in \bigcup_{m \in \mathbb{N}} \left\{ (\varrho, \varrho) \mid \varrho \in s\text{-LOC}_A^{(m)}[Q]_{\text{R}_A} \right\} \right\} \\
&= s\text{-LOC}_A [Q]_{\text{R}_A}.
\end{aligned}$$

Employing Corollary 5.12 instead of Theorem 5.10 in the previous calculation, one may replace the operator  $\text{Inv}_A \text{Pol}_A^{(\leq s)}$  by  $\text{Inv}_A \text{Pol}_A^{(0,s)}$ , and  $\text{Inv}_A \text{Pol}_A^{(\leq s)}$  by  $\text{Inv}_A \text{Pol}_A^{(0,s)}$ , respectively, in the manipulations above.

Due to inapplicability of Lemma 6.16 for  $s = 0$ , this case needs a manual proof. Clearly, we have  $0\text{-LOC}_A [Q]_{\text{R}_A} = \{ \sigma \in \text{R}_A \mid \exists \varrho \in [Q]_{\text{R}_A} : \sigma \supseteq \varrho \}$  and  $\text{Inv}_A \text{Pol}_A^{(0)} Q = \text{Inv}_A \left\{ c_a^{(0)} \mid \forall \varrho \in Q : (a, \dots, a) \in \varrho \right\} = \{ \sigma \in \text{R}_A \mid \sigma \supseteq \mu \}$ , in which  $\mu := \{ (a, \dots, a) \mid a \in A \wedge \forall \varrho \in Q : (a, \dots, a) \in \varrho \}$  and  $c_a^{(0)}$  denotes the nullary operation with image  $\{a\}$ . It is easy to see that  $\mu \in [Q]_{\text{R}_A}$ , namely, for  $\sigma \in \text{R}_A$ , let  $\beta : \text{ar}(\sigma) \rightarrow 1$  and  $\alpha_\varrho : \text{ar}(\varrho) \rightarrow 1$  for  $\varrho \in Q$  be the unique constant mappings, then  $\mu = \bigwedge_{(\alpha_\varrho)_{\varrho \in Q}}^\beta (\varrho)_{\varrho \in Q} \in [Q]_{\text{R}_A}$ . This proves the inclusion  $\text{Inv}_A \text{Pol}_A^{(0)} Q \subseteq 0\text{-LOC}_A [Q]_{\text{R}_A}$ . The converse is simple: if  $\sigma \in \text{R}_A$  includes some  $\varrho \in [Q]_{\text{R}_A}$ , and  $c_a^{(0)} \in \text{Pol}_A^{(0)} Q = \text{Pol}_A^{(0)} [Q]_{\text{R}_A}$ , then  $(a, \dots, a) \in \varrho \subseteq \sigma$ . As this holds for all constants in  $\text{Pol}_A^{(0)} Q$ , we obtain  $\sigma \in \text{Inv}_A \text{Pol}_A^{(0)} Q$ .  $\square$

The previous theorem directly entails Theorem 3.20 of [3, p. 31]:

**Corollary 6.18.** *For  $Q \subseteq \text{R}_A$  we have  $\text{LOC}_A [Q]_{\text{R}_A} = \text{Inv}_A \text{Pol}_A Q$ .*

*Proof.* By definition of the operator  $\text{LOC}_A$  we have for  $Q \subseteq \text{R}_A$ :

$$\begin{aligned}
\text{LOC}_A [Q]_{\text{R}_A} &= \bigcap_{s \in \mathbb{N}} s\text{-LOC}_A [Q]_{\text{R}_A} \\
&= \bigcap_{s \in \mathbb{N}} \text{Inv}_A \text{Pol}_A^{(\leq s)} Q = \text{Inv}_A \bigcup_{s \in \mathbb{N}} \text{Pol}_A^{(\leq s)} Q = \text{Inv}_A \text{Pol}_A Q. \quad \square
\end{aligned}$$

The following evident observation will be needed for the next corollaries.

**Lemma 6.19.** *We have  $\text{Pol}_A \{\emptyset\} = \text{O}_A \setminus \text{O}_A^{(0)}$  and thereby  $\text{Pol}_A^{(s)} \{\emptyset\} = \text{O}_A^{(s)}$  whenever  $s \in \mathbb{N}_+$ ; therefore,  $\text{Pol}_A^{(s)} (Q \cup \{\emptyset\}) = \text{Pol}_A^{(s)} Q$  for any  $Q \subseteq \text{R}_A$ .*

Similarly as in Corollary 5.13, for sets of relations comprising the empty relation, nullary polymorphisms are not required.

**Corollary 6.20.** *Let  $Q \subseteq \text{R}_A$  and  $s \in \mathbb{N}$ . Then  $s\text{-LOC}_A [Q]_{\text{R}_A} = \text{Inv}_A \text{Pol}_A^{(s)} Q$  if (and, provided that  $s > 0$ , also only if)  $\emptyset \in \text{Inv}_A \text{Pol}_A Q$  (which is true in particular if  $\emptyset \in Q$ ).*

*Proof.* The conditions  $\emptyset \in \text{Inv}_A \text{Pol}_A Q$  and  $\text{Pol}_A Q \subseteq \text{Pol}_A \{\emptyset\} = O_A \setminus O_A^{(0)}$  (cf. Lemma 6.19 above) are equivalent; moreover, the latter one holds if and only if  $\text{Pol}_A^{(0)} Q = \emptyset$ . Combining this with the equality from Theorem 6.17 yields  $s\text{-LOC}_A [Q]_{R_A} = \text{Inv}_A \text{Pol}_A^{(s)} Q$ . Conversely, if we assume this equality and suppose  $s > 0$ , which entails  $\text{Pol}_A^{(s)} Q \subseteq O_A \setminus O_A^{(0)}$ , then via Theorem 6.17 we get

$$\begin{aligned} \emptyset \in \text{Inv}_A (O_A \setminus O_A^{(0)}) &\subseteq \text{Inv}_A \text{Pol}_A^{(s)} Q = s\text{-LOC}_A [Q]_{R_A} = \text{Inv}_A \text{Pol}_A^{(\leq s)} Q \\ &\subseteq \text{Inv}_A \text{Pol}_A^{(0)} Q. \end{aligned}$$

This is equivalent to  $\text{Pol}_A^{(0)} Q \subseteq \text{Pol}_A \{\emptyset\} = O_A \setminus O_A^{(0)}$ , i.e.  $\text{Pol}_A^{(0)} Q = \emptyset$ .  $\square$

**Corollary 6.21.** *We have  $\text{LOC}_A [Q \cup \{\emptyset\}]_{R_A} = \text{Inv}_A \text{Pol}_A^{(>0)} Q$  for  $Q \subseteq R_A$ . Moreover, for  $s \in \mathbb{N}_+$  the equality  $s\text{-LOC}_A [Q \cup \{\emptyset\}]_{R_A} = \text{Inv}_A \text{Pol}_A^{(s)} Q$  holds.*

*Proof.* We have  $\text{Pol}_A (Q \cup \{\emptyset\}) = \text{Pol}_A Q \cap (O_A \setminus O_A^{(0)}) = \text{Pol}_A^{(>0)} Q$ , applying Lemma 6.19; thus  $\text{LOC}_A [Q \cup \{\emptyset\}]_{R_A} = \text{Inv}_A \text{Pol}_A (Q \cup \{\emptyset\}) = \text{Inv}_A \text{Pol}_A^{(>0)} Q$  by Corollary 6.18. Combining for  $s \in \mathbb{N}_+$  the statements of Corollary 6.20 and Lemma 6.19 yields the remaining claim.  $\square$

Restricting the statement of Corollary 6.21 to sets  $Q \subseteq R_A \setminus R_A^{(0)}$  and intersecting the equalities on both sides with  $R_A \setminus R_A^{(0)}$  yields the characterisations  $\text{LOC}_A ([Q \cup \{\emptyset\}]_{R_A}^{(>0)}) = \text{Inv}_A^{(>0)} \text{Pol}_A^{(>0)} Q$  and, for positive parameters  $s$ ,  $s\text{-LOC}_A ([Q \cup \{\emptyset\}]_{R_A}^{(>0)}) = \text{Inv}_A^{(>0)} \text{Pol}_A^{(s)} Q$ . The closure  $[Q \cup \{\emptyset\}]_{R_A}^{(>0)}$  describes the appropriate notion of generated relational clone (as employed e.g. in [29]) if one does neither consider nullary operations nor relations in connection with Pol-Inv. With the two stated equalities we have therefore established the two main results (see Theorem 4.2, p. 32, and Theorem 3.3, p. 260, respectively) of [29, 28] regarding the relational side of the mentioned GALOIS connection.

## 7. POSSIBLE APPLICATIONS

In the literature the Pol-Inv GALOIS connection has been very successfully employed to discover the structure of the lattice of all clones (e.g. [31, 22, 37, 36]), but it is also fundamentally involved in investigating other problems in algebra and theoretical computer science ([10, 2, 1, 6, 35]). It is to be expected that the theory developed within this article will find similar applications w.r.t. semiclones in the future, especially regarding infinite carrier sets.

In this connection we briefly outline one possible idea, picking up again the topic of topologically closed (proper) transformation semigroups from the previous section. According to Proposition 6.7, for any set  $Q \subseteq \text{Rp}_A$  of relation pairs, the set of all locally closed transformation semigroups  $S \subseteq O_A^{(1)}$  lying properly below  $\text{Polp}_A^{(1)} Q$  can be described as all those sets  $S = \text{Polp}_A^{(1)} \Sigma$  ( $\Sigma \subseteq \text{Rp}_A$ ) satisfying  $\text{Polp}_A^{(1)} \Sigma \subsetneq \text{Polp}_A^{(1)} Q$ . If  $S$  is a maximal member of this collection with respect to inclusion, then  $Q \subseteq \text{Invp}_A \text{Polp}_A^{(1)} Q \subsetneq \text{Invp}_A S$ . One may take any pair  $(\varrho, \varrho') \in \text{Invp}_A S \setminus \text{Invp}_A \text{Polp}_A^{(1)} Q$  and obtains that

$S = \text{Pol}_A^{(1)} \text{Invp}_A S \subseteq \text{Pol}_A^{(1)} (Q \cup \{(\varrho, \varrho')\}) \subsetneq \text{Pol}_A^{(1)} Q$ , which by maximality of  $S$  entails that  $S = \text{Pol}_A^{(1)} (Q \cup \{(\varrho, \varrho')\})$ . In case that  $\text{Pol}_A^{(1)} Q$  is a monoid, i.e.  $Q$  contains only relation pairs with identical components and  $\text{Pol}_A Q$  is a real clone, then one may also be interested in the maximal locally closed proper transformation semigroups below it. This additional requirement enforces that the pair  $(\varrho, \varrho')$  one had to add above even has to be a proper relation pair, i.e.  $\varrho' \subsetneq \varrho$ .

In a similar way all maximal locally closed (or  $s$ -locally closed) (possibly proper) semiclones (or  $s$ -locally closed transformation semigroups) below one specific structure of the respective sort can be described by preserving one additional relation pair. It is plausible that for certain sets  $Q$  a complete characterisation in analogy to [31] can be attempted. Furthermore, on infinite carrier sets, the machinery developed in this paper can also be useful to reveal counterexamples, e.g. structures having no maximal proper (locally or  $s$ -locally closed) substructures below them. It is, for example, not hard to prove for  $Q = \emptyset$  that proper semigroups of the form  $\text{Pol}_A^{(1)} \{(\varrho, \varrho')\}$  with  $\varrho' \subsetneq \varrho \subseteq A$  can never be maximal among all locally closed proper transformation semigroups on any at least two-element set  $A$ .

The author is, moreover, confident that a generalisation of the presented theory to categories with finite powers is possible along the lines of [23], where a similar project has been realised for clones and the  $\text{Pol}$ - $\text{Inv}$  GALOIS connection (at the same time dualising the involved notions, which is not in our focus). Most of our results do not impose any restrictions on the carrier set, i.e. the particular object of the category of sets the GALOIS theory is based on. Therefore, the main theorems of this article could be a guideline and used to hint at what form of results to expect in the general setting. Once such a generalisation has been established, the corresponding results can be instantiated in any category of interest, as long as it has finite powers, for instance, in that of topological spaces. In this way, it may be possible to perform similar investigations as sketched above also for transformation semigroups consisting of continuous functions.

## REFERENCES

- [1] Libor Barto, *The dichotomy for conservative constraint satisfaction problems revisited*, 26th Annual IEEE Symposium on Logic in Computer Science—LICS 2011, IEEE Computer Soc., Los Alamitos, CA, 2011, pp. 301–310. MR 2858901
- [2] Libor Barto and Marcin Kozik, *Absorbing subalgebras, cyclic terms, and the constraint satisfaction problem*, Log. Methods Comput. Sci. **8** (2012), no. 1, 1:07, 27, 10.2168/LMCS-8(1:7)2012. MR 2893395
- [3] Mike Behrisch, *Clones with nullary operations*, Proceedings of the Workshop on Algebra, Coalgebra and Topology (WACT 2013) (John Power and Cai Wingfield, eds.), Electron. Notes Theor. Comput. Sci., vol. 303, Elsevier Sci. B. V., Amsterdam, March 2014, 10.1016/j.entcs.2014.02.002, pp. 3–35.
- [4] Manuel Bodirsky and Michael Pinsker, *Minimal functions on the random graph*, Israel J. Math. **200** (2014), no. 1, 251–296, 10.1007/s11856-014-1042-y. MR 3219579
- [5] ———, *Topological Birkhoff*, Trans. Amer. Math. Soc. (2014), 23, 10.1090/S0002-9947-2014-05975-8.
- [6] Manuel Bodirsky, Michael Pinsker, and András Pongrácz, *Reconstructing the topology of clones*, CoRR **abs/1312.7699** (2014), 1–30.
- [7] V. G. Bodnarčuk, Lev Arkadevič Kalužnin, Victor N. Kotov, and Boris A. Romov, *Galois theory for Post algebras. I*, Cybernetics **5** (1969), no. 3, 243–252 (English), 10.1007/BF01070906.



- [8] ———, *Galois theory for Post algebras. II*, Cybernetics **5** (1969), no. 5, 531–539 (English), 10.1007/BF01267873.
- [9] ———, *Теория Галуа для алгебр Поста. I, II [Galois theory for Post algebras. I, II]*, Kibernetika (Kiev) **5** (1969), no. 3, 1–10; *ibid.* 1969, no. 5, 1–9. MR 0300895 (46 #55)
- [10] Andrei A. Bulatov, *A dichotomy theorem for constraint satisfaction problems on a 3-element set*, J. ACM **53** (2006), no. 1, 66–120, 10.1145/1120582.1120584. MR 2212000 (2006m:68045)
- [11] Miguel Couceiro, *On Galois connections between external operations and relational constraints: arity restrictions and operator decompositions*, Acta Sci. Math. (Szeged) **72** (2006), no. 1–2, 15–35. MR 2249236 (2007d:08001)
- [12] Miguel Couceiro and Stephan Foldes, *On closed sets of relational constraints and classes of functions closed under variable substitutions*, Algebra Universalis **54** (2005), no. 2, 149–165, 10.1007/s00012-005-1933-1. MR 2217633 (2006m:08002)
- [13] Miguel Couceiro and Erkki Lehtonen, *Galois theory for sets of operations closed under permutation, cylindrification, and composition*, Algebra Universalis **67** (2012), no. 3, 273–297, 10.1007/s00012-012-0184-1. MR 2910127
- [14] David Geiger, *Closed systems of functions and predicates*, Pacific J. Math. **27** (1968), no. 1, 95–100, online available at <http://projecteuclid.org/euclid.pjm/1102985564>. MR 0234893 (38 #3207)
- [15] Martin Goldstern and Michael Pinsker, *A survey of clones on infinite sets*, Algebra Universalis **59** (2008), no. 3–4, 365–403, 10.1007/s00012-008-2100-2. MR 2470587 (2009j:08007)
- [16] Walter Harnau, *Ein verallgemeinerter Relationen- und ein modifizierter Superpositionsbegriff für die Algebra der mehrwertigen Logik. [A generalised notion of relation and a modified notion of superposition for the algebra of multiple-valued logic.]*, Habilitation thesis, Universität Rostock, September 1983, pp. VIII+172.
- [17] ———, *Ein verallgemeinerter Relationenbegriff für die Algebra der mehrwertigen Logik. I. Grundlagen [A generalised notion of relation for the algebra of multiple-valued logic. I. Foundations.]*, Rostock. Math. Kolloq. **28** (1985), 5–17. MR 837174 (88a:03156)
- [18] ———, *Ein verallgemeinerter Relationenbegriff für die Algebra der mehrwertigen Logik. II. Relationenpaare [A generalised notion of relation for the algebra of multiple-valued logic. II. Pairs of relations.]*, Rostock. Math. Kolloq. **31** (1987), 11–20. MR 912673 (90b:03093a)
- [19] ———, *Ein verallgemeinerter Relationenbegriff für die Algebra der mehrwertigen Logik. III. Beweis [A generalised notion of relation for the algebra of multiple-valued logic. III. Proof.]*, Rostock. Math. Kolloq. **32** (1987), 15–24. MR 937662 (90b:03093b)
- [20] Sergej Vsevolodovič Jablonskij, *Функциональные построения в  $k$ -значной логике [Functional constructions in  $k$ -valued logic]*, Trudy Mat. Inst. Steklov. **51** (1958), 5–142. MR 0104578 (21 #3331)
- [21] Sergej Vsevolodovič Jablonskij, G. P. Gavrillov, and Valerij Borisovič Kudrjavcev, *Функции алгебры логики и классы Поста [Functions of algebraic logic and Post classes]*, Izdat. “Nauka”, Moscow, 1966. MR 0215649 (35 #6489)
- [22] Ju. I. Janov and Albert Abramovič Mučnik, *О существовании  $k$ -значных замкнутых классов, не имеющих конечного базиса [Existence of  $k$ -valued closed classes without a finite basis]*, Dokl. Akad. Nauk SSSR **127** (1959), no. 1, 44–46. MR 0108458 (21 #7174)
- [23] Sebastian Kerkhoff, *A general Galois theory for operations and relations in arbitrary categories*, Algebra Universalis **68** (2012), no. 3–4, 325–352, 10.1007/s00012-012-0209-9. MR 3029960
- [24] Dietlinde Lau, *Function algebras on finite sets*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006, A basic course on many-valued logic and clone theory. MR 2254622 (2007m:03001)
- [25] Erkki Lehtonen, *Closed classes of functions, generalized constraints, and clusters*, Algebra Universalis **63** (2010), no. 2–3, 203–234 (English), 10.1007/s00012-010-0071-6. MR 2728136 (2012b:08006)
- [26] Anatolij Ivanovič Maļcev, *Итеративные алгебры и многообразия Поста [Iterative algebras and Post varieties]*, Algebra i Logika Sem. **5** (1966), no. 2, 5–24. MR 0207609 (34 #7424)

- [27] Nicholas Pippenger, *Galois theory for minors of finite functions*, Discrete Math. **254** (2002), no. 1-3, 405–419, 10.1016/S0012-365X(01)00297-7. MR 1910121 (2003i:06016)
- [28] Reinhard Pöschel, *Concrete representation of algebraic structures and a general Galois theory*, Contributions to general algebra (Proc. Klagenfurt Conf., Klagenfurt, 1978), Heyn, Klagenfurt, 1979, pp. 249–272. MR 537425 (80h:08008)
- [29] ———, *A general Galois theory for operations and relations and concrete characterization of related algebraic structures*, Report 1980, vol. 1, Akademie der Wissenschaften der DDR Institut für Mathematik, Berlin, 1980, With German and Russian summaries. MR 568709 (81h:08003)
- [30] Reinhard Pöschel and Lev Arkadevič Kalužnin, *Funktionen- und Relationenalgebren*, Mathematische Monographien [Mathematical Monographs], vol. 15, VEB Deutscher Verlag der Wissenschaften, Berlin, 1979, Ein Kapitel der diskreten Mathematik. [A chapter in discrete mathematics]. MR 543839 (81f:03075)
- [31] Ivo G. Rosenberg, *Über die funktionale Vollständigkeit in den mehrwertigen Logiken. Struktur der Funktionen von mehreren Veränderlichen auf endlichen Mengen*, vol. 80, Rozprawy Československé Akademie Věd: Řada matematických a přírodních věd, no. 4, Academia, 1970. MR 0292647 (45 #1732)
- [32] Jürgen Schmidt, *Clones and semiclones of operations*, Universal algebra (Esztergom, 1977), Colloq. Math. Soc. János Bolyai, vol. 29, North-Holland, Amsterdam-New York, 1982, pp. 705–723. MR 660904 (83m:08005)
- [33] László Szabó, *Concrete representation of related structures of universal algebras. I*, Acta Sci. Math. (Szeged) **40** (1978), no. 1-2, 175–184. MR 0480264 (58 #443)
- [34] Ágnes Szendrei, *Clones in universal algebra*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 99, Presses de l'Université de Montréal, Montreal, QC, 1986. MR 859550 (87m:08005)
- [35] Edith Vargas, *Clausal relations and c-clones*, Discuss. Math. Gen. Algebra Appl. **30** (2010), no. 2, 147–171, [10.7151/dmgaa.1167](https://doi.org/10.7151/dmgaa.1167). MR 2814091 (2012b:08001)
- [36] Dmitriy N. Zhuk, *The lattice of the clones of self-dual functions in three-valued logic*, Proceedings of the symposium held in Tuusula, May 23–25, 2011, 41st IEEE International Symposium on Multiple-Valued Logic ISMVL 2011, IEEE Computer Society, Los Alamitos, CA, May 2011, pp. 193–197.
- [37] ———, *The cardinality of the set of all clones containing a given minimal clone on three elements*, Algebra Universalis **68** (2012), no. 3-4, 295–320 (English), 10.1007/s00012-012-0207-y. MR 3029958

INSTITUT FÜR COMPUTERSPRACHEN, TECHNISCHE UNIVERSITÄT WIEN, A-1040 VIENNA,  
AUSTRIA

*E-mail address:* behrisch@logic.at